Section 3.6 Quadratic Inequalities

Essential Question How can you solve a quadratic inequality?

Exploration 1 Solving a Quadratic Inequality

Work with a partner. The graphing calculator screen shows the graph of
\[ f(x) = x^2 + 2x - 3. \]

Explain how you can use the graph to solve the inequality
\[ x^2 + 2x - 3 \leq 0. \]

Then solve the inequality.

Exploration 2 Solving Quadratic Inequalities

Work with a partner. Match each inequality with the graph of its related quadratic function. Then use the graph to solve the inequality.

a. \( x^2 - 3x + 2 > 0 \)  
b. \( x^2 - 4x + 3 \leq 0 \)  
c. \( x^2 - 2x - 3 < 0 \)
d. \( x^2 + x - 2 \geq 0 \)  
e. \( x^2 - x - 2 < 0 \)  
f. \( x^2 - 4 > 0 \)

Communicate Your Answer

3. How can you solve a quadratic inequality?
4. Explain how you can use the graph in Exploration 1 to solve each inequality. Then solve each inequality.
   - a. \( x^2 + 2x - 3 > 0 \)
   - b. \( x^2 + 2x - 3 < 0 \)
   - c. \( x^2 + 2x - 3 \geq 0 \)
What You Will Learn

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in one variable.

Graphing Quadratic Inequalities in Two Variables

A quadratic inequality in two variables can be written in one of the following forms, where \(a\), \(b\), and \(c\) are real numbers and \(a \neq 0\).

\[
\begin{align*}
  y &< ax^2 + bx + c \\
  y &> ax^2 + bx + c \\
  y &\leq ax^2 + bx + c \\
  y &\geq ax^2 + bx + c
\end{align*}
\]

The graph of any such inequality consists of all solutions \((x, y)\) of the inequality.

Previously, you graphed linear inequalities in two variables. You can use a similar procedure to graph quadratic inequalities in two variables.

Graphing a Quadratic Inequality in Two Variables

To graph a quadratic inequality in one of the forms above, follow these steps.

**Step 1** Graph the parabola with the equation \(y = ax^2 + bx + c\). Make the parabola **dashed** for inequalities with < or > and **solid** for inequalities with \(\leq\) or \(\geq\).

**Step 2** Test a point \((x, y)\) inside the parabola to determine whether the point is a solution of the inequality.

**Step 3** Shade the region inside the parabola if the point from Step 2 is a solution. Shade the region outside the parabola if it is not a solution.

**EXAMPLE 1** Graphing a Quadratic Inequality in Two Variables

Graph \(y < -x^2 - 2x - 1\).

**SOLUTION**

**Step 1** Graph \(y = -x^2 - 2x - 1\). Because the inequality symbol is <, make the parabola dashed.

**Step 2** Test a point inside the parabola, such as \((0, -3)\).

\[
\begin{align*}
  y &< -x^2 - 2x - 1 \\
  -3 &< -0^2 - 2(0) - 1 \\
  -3 &< -1 \quad \checkmark
\end{align*}
\]

So, \((0, -3)\) is a solution of the inequality.

**Step 3** Shade the region inside the parabola.
A manila rope used for rappelling down a cliff can safely support a weight \(W\) (in pounds) provided

\[ W \leq 1480d^2 \]

where \(d\) is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

**SOLUTION**

Graph \(W = 1480d^2\) for nonnegative values of \(d\). Because the inequality symbol is \(\leq\), make the parabola solid. Test a point inside the parabola, such as \((1, 3000)\).

\[ W \leq 1480d^2 \]

\[ 3000 \leq 1480(1)^2 \]

\[ 3000 \not\leq 1480 \]

Because \((1, 3000)\) is not a solution, shade the region outside the parabola. The shaded region represents weights that can be supported by ropes with various diameters.

Graphing a **system** of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the system. Then identify the region in the coordinate plane common to all of the graphs. This region is called the **graph of the system**.

**EXAMPLE 3**  
**Graphing a System of Quadratic Inequalities**

Graph the system of quadratic inequalities.

\[ y < -x^2 + 3 \quad \text{Inequality 1} \]
\[ y \geq x^2 + 2x - 3 \quad \text{Inequality 2} \]

**SOLUTION**

**Step 1** Graph \(y < -x^2 + 3\). The graph is the red region inside (but not including) the parabola \(y = -x^2 + 3\).

**Step 2** Graph \(y \geq x^2 + 2x - 3\). The graph is the blue region inside and including the parabola \(y = x^2 + 2x - 3\).

**Step 3** Identify the **purple region** where the two graphs overlap. This region is the graph of the system.

**Monitoring Progress**

Graph the inequality.

1. \(y \geq x^2 + 2x - 8\)
2. \(y \leq 2x^2 - x - 1\)
3. \(y > -x^2 + 2x + 4\)
4. Graph the system of inequalities consisting of \(y \leq -x^2\) and \(y > x^2 - 3\).
Solving Quadratic Inequalities in One Variable

A **quadratic inequality in one variable** can be written in one of the following forms, where \(a, b,\) and \(c\) are real numbers and \(a \neq 0\).

\[ax^2 + bx + c < 0\quad ax^2 + bx + c > 0\quad ax^2 + bx + c \leq 0\quad ax^2 + bx + c \geq 0\]

You can solve quadratic inequalities using algebraic methods or graphs.

### Example 4

**Solving a Quadratic Inequality Algebraically**

Solve \(x^2 - 3x - 4 < 0\) algebraically.

**SOLUTION**

First, write and solve the equation obtained by replacing \(<\) with \(=\).

\[x^2 - 3x - 4 = 0\]

Write the related equation.

\[(x - 4)(x + 1) = 0\]

Factor.

\[x = 4\quad \text{or}\quad x = -1\]

Zero-Product Property

The numbers \(-1\) and \(4\) are the **critical values** of the original inequality. Plot \(-1\) and \(4\) on a number line, using open dots because the values do not satisfy the inequality. The critical \(x\)-values partition the number line into three intervals. Test an \(x\)-value in each interval to determine whether it satisfies the inequality.

\[\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Interval} & \text{Test} & \text{Result} \\
\hline
(-\infty, -1) & x = -2 & > 0 & \checkmark & \text{No solution} \\
(-1, 4) & x = 0 & < 0 & \checkmark & \\
(4, \infty) & x = 5 & > 0 & & \text{No solution} \\
\end{array}\]

\(-2)^2 - 3(-2) - 4 = 6 \neq 0\quad 0^2 - 3(0) - 4 = -4 < 0\quad 5^2 - 3(5) - 4 = 6 \neq 0\)

\[\checkmark\]

So, the solution is \(-1 < x < 4\).

Another way to solve \(ax^2 + bx + c < 0\) is to first graph the related function \(y = ax^2 + bx + c\). Then, because the inequality symbol is \(<\), identify the \(x\)-values for which the graph lies \text{below} the \(x\)-axis. You can use a similar procedure to solve quadratic inequalities that involve \(\leq\), \(>\), or \(\geq\).

### Example 5

**Solving a Quadratic Inequality by Graphing**

Solve \(3x^2 - x - 5 \geq 0\) by graphing.

**SOLUTION**

The solution consists of the \(x\)-values for which the graph of \(y = 3x^2 - x - 5\) lies on or above the \(x\)-axis. Find the \(x\)-intercepts of the graph by letting \(y = 0\) and using the Quadratic Formula to solve \(0 = 3x^2 - x - 5\) for \(x\).

\[x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)}\]

\[x = \frac{1 \pm \sqrt{61}}{6}\]

Simplify.

The solutions are \(x \approx -1.14\) and \(x \approx 1.47\). Sketch a parabola that opens up and has \(-1.14\) and \(1.47\) as \(x\)-intercepts. The graph lies on or above the \(x\)-axis to the left of (and including) \(x = -1.14\) and to the right of (and including) \(x = 1.47\).

\[\checkmark\]

The solution of the inequality is approximately \(x \leq -1.14\) or \(x \geq 1.47\).
A rectangular parking lot must have a perimeter of 440 feet and an area of at least 8000 square feet. Describe the possible lengths of the parking lot.

**SOLUTION**

1. **Understand the Problem** You are given the perimeter and the minimum area of a parking lot. You are asked to determine the possible lengths of the parking lot.

2. **Make a Plan** Use the perimeter and area formulas to write a quadratic inequality describing the possible lengths of the parking lot. Then solve the inequality.

3. **Solve the Problem** Let $l$ represent the length (in feet) and let $w$ represent the width (in feet) of the parking lot.

   \begin{align*}
   \text{Perimeter} &= 440 \\
   2l + 2w &= 440 \\
   \text{Area} &\geq 8000 \\
   lw &\geq 8000
   \end{align*}

   Solve the perimeter equation for $w$ to obtain $w = \frac{220 - l}{2}$.

   Substitute this into the area inequality to obtain a quadratic inequality in one variable.

   \begin{align*}
   l\left(\frac{220 - l}{2}\right) &\geq 8000 \\
   \frac{-l^2 + 220l}{2} &\geq 8000
   \end{align*}

   Write the area inequality.

   \begin{align*}
   \frac{-l^2 + 220l}{2} &\geq 8000 \\
   -l^2 + 220l - 8000 &\geq 0
   \end{align*}

   Distributive Property

   Use a graphing calculator to find the $l$-intercepts of $y = -l^2 + 220l - 8000$.

   The $l$-intercepts are $l \approx 45.97$ and $l \approx 174.03$. The solution consists of the $l$-values for which the graph lies on or above the $l$-axis. The graph lies on or above the $l$-axis when $45.97 \leq l \leq 174.03$.

   So, the approximate length of the parking lot is at least 46 feet and at most 174 feet.

4. **Look Back** Choose a length in the solution region, such as $l = 100$, and find the width. Then check that the dimensions satisfy the original area inequality.

   \begin{align*}
   2l + 2w &= 440 \\
   l &\geq 8000 \\
   2(100) + 2w &= 440 \\
   100(110) &\geq 8000 \\
   w &= 110 \\
   11,000 &\geq 8000 \checkmark
   \end{align*}

**Monitoring Progress**

Solve the inequality.

5. $2x^2 + 3x \leq 2$  
6. $-3x^2 - 4x + 1 < 0$  
7. $2x^2 + 2 > -5x$

8. **WHAT IF?** In Example 6, the area must be at least 8500 square feet. Describe the possible lengths of the parking lot.
1. **WRITING** Compare the graph of a quadratic inequality in one variable to the graph of a quadratic inequality in two variables.

2. **WRITING** Explain how to solve \(x^2 + 6x - 8 < 0\) using algebraic methods and using graphs.

**Exercises 3.6**

**Vocabulary and Core Concept Check**

In Exercises 3–6, match the inequality with its graph. Explain your reasoning.

3. \(y \leq x^2 + 4x + 3\)
4. \(y > -x^2 + 4x - 3\)
5. \(y < x^2 - 4x + 3\)
6. \(y \geq x^2 + 4x + 3\)

**ERROR ANALYSIS** In Exercises 17 and 18, describe and correct the error in graphing \(y \geq x^2 + 2\).

17. [Graph with incorrect shading]
18. [Graph with incorrect shading]

19. **MODELING WITH MATHEMATICS** A hardwood shelf in a wooden bookcase can safely support a weight \(W\) (in pounds) provided \(W \leq 115x^2\), where \(x\) is the thickness (in inches) of the shelf. Graph the inequality and interpret the solution. (See Example 2.)

20. **MODELING WITH MATHEMATICS** A wire rope can safely support a weight \(W\) (in pounds) provided \(W \leq 8000d^2\), where \(d\) is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

In Exercises 7–14, graph the inequality. (See Example 1.)

7. \(y < x^2\)
8. \(y \geq 4x^2\)
9. \(y > x^2 - 9\)
10. \(y < x^2 + 5\)
11. \(y \leq x^2 + 5x\)
12. \(y \geq -2x^2 + 9x - 4\)
13. \(y > 2(x + 3)^2 - 1\)
14. \(y \leq \left(x - \frac{1}{2}\right)^2 + \frac{5}{2}\)

**ANALYZING RELATIONSHIPS** In Exercises 15 and 16, use the graph to write an inequality in terms of \(f(x)\) so point \(P\) is a solution.

15. [Graph with point \(P\)]
16. [Graph with point \(P\) and function \(y = f(x)\)]

**In Exercises 21–26, graph the system of quadratic inequalities. (See Example 3.)**

21. \(y \geq 2x^2\)
22. \(y > -5x^2\)
23. \(y \leq -x^2 + 4x - 4\)
24. \(y \geq x^2 - 4\)
25. \(y \geq 2x^2 + x - 5\)
26. \(y \geq x^2 - 3x - 6\)
27. \(y < -x^2 + 5x + 10\)
28. \(y > x^2 + 7x + 6\)
In Exercises 27–34, solve the inequality algebraically.  
(See Example 4.)

27. \(4x^2 < 25\)

28. \(x^2 + 10x + 9 < 0\)

29. \(x^2 - 11x \geq -28\)

30. \(3x^2 - 13x > -10\)

31. \(2x^2 - 5x - 3 \leq 0\)

32. \(4x^2 + 8x - 21 \geq 0\)

33. \(\frac{1}{2}x^2 - x > 4\)

34. \(-\frac{1}{2}x^2 + 4x \leq 1\)

In Exercises 35–42, solve the inequality by graphing.  
(See Example 5.)

35. \(x^2 - 3x + 1 < 0\)

36. \(x^2 - 4x + 2 > 0\)

37. \(x^2 + 8x > -7\)

38. \(x^2 + 6x < -3\)

39. \(3x^2 - 8 \leq -2x\)

40. \(3x^2 + 5x - 3 < 1\)

41. \(\frac{1}{3}x^2 + 2x \geq 2\)

42. \(\frac{3}{4}x^2 + 4x \geq 3\)

43. **DRAWING CONCLUSIONS** Consider the graph of the function \(f(x) = ax^2 + bx + c\).

a. What are the solutions of \(ax^2 + bx + c < 0\)?

b. What are the solutions of \(ax^2 + bx + c > 0\)?

c. The graph of \(g\) represents a reflection in the \(x\)-axis of the graph of \(f\). For which values of \(x\) is \(g(x)\) positive?

44. **MODELING WITH MATHEMATICS** A rectangular fountain display has a perimeter of 400 feet and an area of at least 9100 feet. Describe the possible widths of the fountain.  
(See Example 6.)

45. **MODELING WITH MATHEMATICS** The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by \(y = -0.00211x^2 + 1.06x\), where \(x\) is the distance (in meters) from the left pylons and \(y\) is the height (in meters) of the arch above the water. For what distances \(x\) is the arch above the road?

46. **PROBLEM SOLVING** The number \(T\) of teams that have participated in a robot-building competition for high-school students over a recent period of time \(x\) (in years) can be modeled by

\[T(x) = 17.155x^2 + 193.68x + 235.81, 0 \leq x \leq 6.\]

After how many years is the number of teams greater than 1000? Justify your answer.

47. **PROBLEM SOLVING** A study found that a driver’s reaction time \(A(x)\) to audio stimuli and his or her reaction time \(V(x)\) to visual stimuli (both in milliseconds) can be modeled by

\[A(x) = 0.0051x^2 - 0.319x + 15, 16 \leq x \leq 70\]

\[V(x) = 0.005x^2 - 0.23x + 22, 16 \leq x \leq 70\]

where \(x\) is the age (in years) of the driver.

a. Write an inequality that you can use to find the \(x\)-values for which \(A(x)\) is less than \(V(x)\).

b. Use a graphing calculator to solve the inequality \(A(x) < V(x)\). Describe how you used the domain \(16 \leq x \leq 70\) to determine a reasonable solution.

c. Based on your results from parts (a) and (b), do you think a driver would react more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.
48. **HOW DO YOU SEE IT?** The graph shows a system of quadratic inequalities.

![Graph of quadratic inequalities](image)

a. Identify two solutions of the system.

b. Are the points (1, −2) and (5, 6) solutions of the system? Explain.

c. Is it possible to change the inequality symbol(s) so that one, but not both of the points, is a solution of the system? Explain.

49. **MODELING WITH MATHEMATICS** The length $L$ (in millimeters) of the larvae of the black porgy fish can be modeled by

$$L(x) = 0.00170 x^2 + 0.145 x + 2.35, \quad 0 \leq x \leq 40$$

where $x$ is the age (in days) of the larvae. Write and solve an inequality to find at what ages a larva’s length tends to be greater than 10 millimeters. Explain how the given domain affects the solution.

50. **MAKING AN ARGUMENT** You claim the system of inequalities below, where $a$ and $b$ are real numbers, has no solution. Your friend claims the system will always have at least one solution. Who is correct? Explain.

$$y < (x + a)^2$$

$$y < (x + b)^2$$

51. **MATHEMATICAL CONNECTIONS** The area $A$ of the region bounded by a parabola and a horizontal line can be modeled by $A = \frac{2}{3}bh$, where $b$ and $h$ are as defined in the diagram. Find the area of the region determined by each pair of inequalities.

![Diagram of area calculation](image)

a. $y \leq -x^2 + 4x$

b. $y \geq x^2 - 4x - 5$

52. **THOUGHT PROVOKING** Draw a company logo that is created by the intersection of two quadratic inequalities. Justify your answer.

53. **REASONING** A truck that is 11 feet tall and 7 feet wide is traveling under an arch. The arch can be modeled by $y = -0.0625x^2 + 1.25x + 5.75$, where $x$ and $y$ are measured in feet.

a. Will the truck fit under the arch? Explain.

b. What is the maximum width that a truck 11 feet tall can have and still make it under the arch?

c. What is the maximum height that a truck 7 feet wide can have and still make it under the arch?