4.2 Adding, Subtracting, and Multiplying Polynomials

COMMON CORE

Learning Standards
HSA-APR.A.1
HSA-APR.C.4
HSA-APR.C.5

Essential Question

How can you cube a binomial?

EXPLORATION 1 Cubing Binomials

Work with a partner. Find each product. Show your steps.

a. $(x + 1)^3 = (x + 1)(x + 1)^2$ — Rewrite as a product of first and second powers.

$= (x + 1)$ ______ — Multiply second power.

$=$ ______ — Multiply binomial and trinomial.

$=$ ______ — Write in standard form, $ax^3 + bx^2 + cx + d$.

b. $(a + b)^3 = (a + b)(a + b)^2$ — Rewrite as a product of first and second powers.

$= (a + b)$ ______ — Multiply second power.

$=$ ______ — Multiply binomial and trinomial.

$=$ ______ — Write in standard form.

c. $(x - 1)^3 = (x - 1)(x - 1)^2$ — Rewrite as a product of first and second powers.

$= (x - 1)$ ______ — Multiply second power.

$=$ ______ — Multiply binomial and trinomial.

$=$ ______ — Write in standard form.

d. $(a - b)^3 = (a - b)(a - b)^2$ — Rewrite as a product of first and second powers.

$= (a - b)$ ______ — Multiply second power.

$=$ ______ — Multiply binomial and trinomial.

$=$ ______ — Write in standard form.

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Generalizing Patterns for Cubing a Binomial

Work with a partner.

a. Use the results of Exploration 1 to describe a pattern for the coefficients of the terms when you expand the cube of a binomial. How is your pattern related to Pascal's Triangle, shown at the right?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

b. Use the results of Exploration 1 to describe a pattern for the exponents of the terms in the expansion of a cube of a binomial.

c. Explain how you can use the patterns you described in parts (a) and (b) to find the product $(2x - 3)^3$. Then find this product.

Communicate Your Answer

3. How can you cube a binomial?

4. Find each product.

a. $(x + 2)^3$ **b.** $(x - 2)^3$ **c.** $(2x - 3)^3$

d. $(x - 3)^3$ **e.** $(-2x + 3)^3$ **f.** $(3x - 5)^3$

4.2 Lesson

Core Vocabulary

Pascal's Triangle, *p. 169*

Previous
like terms
identity

What You Will Learn

- Add and subtract polynomials.
- Multiply polynomials.
- Use Pascal's Triangle to expand binomials.

Adding and Subtracting Polynomials

Recall that the set of integers is *closed* under addition and subtraction because every sum or difference results in an integer. To add or subtract polynomials, you add or subtract the coefficients of like terms. Because adding or subtracting polynomials results in a polynomial, the set of polynomials is closed under addition and subtraction.

EXAMPLE 1 Adding Polynomials Vertically and Horizontally

a. Add $3x^3 + 2x^2 - x - 7$ and $x^3 - 10x^2 + 8$ in a vertical format.

b. Add $9y^3 + 3y^2 - 2y + 1$ and $-5y^2 + y - 4$ in a horizontal format.

SOLUTION

a.

$$\begin{array}{rrrrr} & 3x^3 + & 2x^2 & - x & - 7 \\ + & x^3 - & 10x^2 & & + 8 \\ \hline & 4x^3 - & 8x^2 & - x & + 1 \end{array}$$

b.

$$\begin{aligned}(9y^3 + 3y^2 - 2y + 1) + (-5y^2 + y - 4) &= 9y^3 + 3y^2 - 5y^2 - 2y + y + 1 - 4 \\ &= 9y^3 - 2y^2 - y - 3\end{aligned}$$

To subtract one polynomial from another, add the opposite. To do this, change the sign of each term of the subtracted polynomial and then add the resulting like terms.

COMMON ERROR

A common mistake is to forget to change signs correctly when subtracting one polynomial from another. Be sure to add the opposite of *every* term of the subtracted polynomial.

EXAMPLE 2 Subtracting Polynomials Vertically and Horizontally

a. Subtract $2x^3 + 6x^2 - x + 1$ from $8x^3 - 3x^2 - 2x + 9$ in a vertical format.

b. Subtract $3z^2 + z - 4$ from $2z^2 + 3z$ in a horizontal format.

SOLUTION

a. Align like terms, then add the opposite of the subtracted polynomial.

$$\begin{array}{r} 8x^3 - 3x^2 - 2x + 9 \\ -\ (2x^3 + 6x^2 - x + 1) \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{rr} & 8x^3 - 3x^2 - 2x + 9 \\ + & -2x^3 - 6x^2 + x - 1 \\ \hline & 6x^3 - 9x^2 - x + 8 \end{array}$$

b. Write the opposite of the subtracted polynomial, then add like terms.

$$\begin{aligned}(2z^2 + 3z) - (3z^2 + z - 4) &= 2z^2 + 3z - 3z^2 - z + 4 \\ &= -z^2 + 2z + 4\end{aligned}$$

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Find the sum or difference.

1. $(2x^2 - 6x + 5) + (7x^2 - x - 9)$

2. $(3t^3 + 8t^2 - t - 4) - (5t^3 - t^2 + 17)$

Multiplying Polynomials

To multiply two polynomials, you multiply each term of the first polynomial by each term of the second polynomial. As with addition and subtraction, the set of polynomials is closed under multiplication.

EXAMPLE 3 Multiplying Polynomials Vertically and Horizontally

a. Multiply $-x^2 + 2x + 4$ and $x - 3$ in a vertical format.

b. Multiply $y + 5$ and $3y^2 - 2y + 2$ in a horizontal format.

REMEMBER

Product of Powers Property

$a^m \cdot a^n = a^{m+n}$

a is a real number and m and n are integers.

SOLUTION

a.

$$\begin{array}{rl}
-x^2 + 2x + 4 & \\
\times \qquad\qquad x - 3 & \\
\hline
3x^2 - 6x - 12 & \text{Multiply } -x^2 + 2x + 4 \text{ by } -3. \\
-x^3 + 2x^2 + 4x \qquad & \text{Multiply } -x^2 + 2x + 4 \text{ by } x. \\
\hline
-x^3 + 5x^2 - 2x - 12 & \text{Combine like terms.}
\end{array}$$

b. $$\begin{aligned}(y + 5)(3y^2 - 2y + 2) &= (y + 5)3y^2 - (y + 5)2y + (y + 5)2 \\ &= 3y^3 + 15y^2 - 2y^2 - 10y + 2y + 10 \\ &= 3y^3 + 13y^2 - 8y + 10\end{aligned}$$

EXAMPLE 4 Multiplying Three Binomials

Multiply $x - 1$, $x + 4$, and $x + 5$ in a horizontal format.

SOLUTION

$$\begin{aligned}(x - 1)(x + 4)(x + 5) &= (x^2 + 3x - 4)(x + 5) \\ &= (x^2 + 3x - 4)x + (x^2 + 3x - 4)5 \\ &= x^3 + 3x^2 - 4x + 5x^2 + 15x - 20 \\ &= x^3 + 8x^2 + 11x - 20\end{aligned}$$

Some binomial products occur so frequently that it is worth memorizing their patterns. You can verify these polynomial identities by multiplying.

COMMON ERROR

In general,

$(a \pm b)^2 \neq a^2 \pm b^2$

and

$(a \pm b)^3 \neq a^3 \pm b^3$.

Core Concept

Special Product Patterns

Sum and Difference	**Example**
$(a + b)(a - b) = a^2 - b^2$	$(x + 3)(x - 3) = x^2 - 9$
Square of a Binomial	**Example**
$(a + b)^2 = a^2 + 2ab + b^2$	$(y + 4)^2 = y^2 + 8y + 16$
$(a - b)^2 = a^2 - 2ab + b^2$	$(2t - 5)^2 = 4t^2 - 20t + 25$
Cube of a Binomial	**Example**
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(z + 3)^3 = z^3 + 9z^2 + 27z + 27$
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(m - 2)^3 = m^3 - 6m^2 + 12m - 8$

EXAMPLE 5 Proving a Polynomial Identity

a. Prove the polynomial identity for the cube of a binomial representing a sum:

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

b. Use the cube of a binomial in part (a) to calculate 11^3.

SOLUTION

a. Expand and simplify the expression on the left side of the equation.

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b)\\ &= (a^2 + 2ab + b^2)(a + b)\\ &= (a^2 + 2ab + b^2)a + (a^2 + 2ab + b^2)b\\ &= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3 \ \checkmark\end{aligned}$$

▶ The simplified left side equals the right side of the original identity. So, the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is true.

b. To calculate 11^3 using the cube of a binomial, note that $11 = 10 + 1$.

$11^3 = (10 + 1)^3$	Write 11 as 10 + 1.
$= 10^3 + 3(10)^2(1) + 3(10)(1)^2 + 1^3$	Cube of a binomial
$= 1000 + 300 + 30 + 1$	Expand.
$= 1331$	Simplify.

REMEMBER

Power of a Product Property

$(ab)^m = a^mb^m$

a and b are real numbers and m is an integer.

EXAMPLE 6 Using Special Product Patterns

Find each product.

a. $(4n + 5)(4n - 5)$ **b.** $(9y - 2)^2$ **c.** $(ab + 4)^3$

SOLUTION

a. $(4n + 5)(4n - 5) = (4n)^2 - 5^2$	Sum and difference
$= 16n^2 - 25$	Simplify.
b. $(9y - 2)^2 = (9y)^2 - 2(9y)(2) + 2^2$	Square of a binomial
$= 81y^2 - 36y + 4$	Simplify.
c. $(ab + 4)^3 = (ab)^3 + 3(ab)^2(4) + 3(ab)(4)^2 + 4^3$	Cube of a binomial
$= a^3b^3 + 12a^2b^2 + 48ab + 64$	Simplify.

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Find the product.

3. $(4x^2 + x - 5)(2x + 1)$

4. $(y - 2)(5y^2 + 3y - 1)$

5. $(m - 2)(m - 1)(m + 3)$

6. $(3t - 2)(3t + 2)$

7. $(5a + 2)^2$

8. $(xy - 3)^3$

9. (a) Prove the polynomial identity for the cube of a binomial representing a difference: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

(b) Use the cube of a binomial in part (a) to calculate 9^3.

Pascal's Triangle

Consider the expansion of the binomial $(a + b)^n$ for whole number values of n. When you arrange the coefficients of the variables in the expansion of $(a + b)^n$, you will see a special pattern called **Pascal's Triangle**. Pascal's Triangle is named after French mathematician Blaise Pascal (1623–1662).

Core Concept

Pascal's Triangle

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a+b)^n$	Binomial Expansion	Pascal's Triangle
0th row	0	$(a+b)^0 =$	1	1
1st row	1	$(a+b)^1 =$	$1a + 1b$	1 1
2nd row	2	$(a+b)^2 =$	$1a^2 + 2ab + 1b^2$	1 2 1
3rd row	3	$(a+b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
4th row	4	$(a+b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
5th row	5	$(a+b)^5 =$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

In general, the nth row in Pascal's Triangle gives the coefficients of $(a + b)^n$. Here are some other observations about the expansion of $(a + b)^n$.

1. An expansion has $n + 1$ terms.
2. The power of a begins with n, decreases by 1 in each successive term, and ends with 0.
3. The power of b begins with 0, increases by 1 in each successive term, and ends with n.
4. The sum of the powers of each term is n.

EXAMPLE 7 Using Pascal's Triangle to Expand Binomials

Use Pascal's Triangle to expand (a) $(x - 2)^5$ and (b) $(3y + 1)^3$.

SOLUTION

a. The coefficients from the fifth row of Pascal's Triangle are 1, 5, 10, 10, 5, and 1.

$$(x - 2)^5 = 1x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + 1(-2)^5$$
$$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

b. The coefficients from the third row of Pascal's Triangle are 1, 3, 3, and 1.

$$(3y + 1)^3 = 1(3y)^3 + 3(3y)^2(1) + 3(3y)(1)^2 + 1(1)^3$$
$$= 27y^3 + 27y^2 + 9y + 1$$

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10. Use Pascal's Triangle to expand (a) $(z + 3)^4$ and (b) $(2t - 1)^5$.

4.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe three different methods to expand $(x + 3)^3$.

2. **WRITING** Is $(a + b)(a - b) = a^2 - b^2$ an identity? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the sum. *(See Example 1.)*

3. $(3x^2 + 4x - 1) + (-2x^2 - 3x + 2)$

4. $(-5x^2 + 4x - 2) + (-8x^2 + 2x + 1)$

5. $(12x^5 - 3x^4 + 2x - 5) + (8x^4 - 3x^3 + 4x + 1)$

6. $(8x^4 + 2x^2 - 1) + (3x^3 - 5x^2 + 7x + 1)$

7. $(7x^6 + 2x^5 - 3x^2 + 9x) + (5x^5 + 8x^3 - 6x^2 + 2x - 5)$

8. $(9x^4 - 3x^3 + 4x^2 + 5x + 7) + (11x^4 - 4x^2 - 11x - 9)$

In Exercises 9–14, find the difference. *(See Example 2.)*

9. $(3x^3 - 2x^2 + 4x - 8) - (5x^3 + 12x^2 - 3x - 4)$

10. $(7x^4 - 9x^3 - 4x^2 + 5x + 6) - (2x^4 + 3x^3 - x^2 + x - 4)$

11. $(5x^6 - 2x^4 + 9x^3 + 2x - 4) - (7x^5 - 8x^4 + 2x - 11)$

12. $(4x^5 - 7x^3 - 9x^2 + 18) - (14x^5 - 8x^4 + 11x^2 + x)$

13. $(8x^5 + 6x^3 - 2x^2 + 10x) - (9x^5 - x^3 - 13x^2 + 4)$

14. $(11x^4 - 9x^2 + 3x + 11) - (2x^4 + 6x^3 + 2x - 9)$

15. **MODELING WITH MATHEMATICS** During a recent period of time, the numbers (in thousands) of males M and females F that attend degree-granting institutions in the United States can be modeled by

$$M = 19.7t^2 + 310.5t + 7539.6$$
$$F = 28t^2 + 368t + 10127.8$$

where t is time in years. Write a polynomial to model the total number of people attending degree-granting institutions. Interpret its constant term.

16. **MODELING WITH MATHEMATICS** A farmer plants a garden that contains corn and pumpkins. The total area (in square feet) of the garden is modeled by the expression $2x^2 + 5x + 4$. The area of the corn is modeled by the expression $x^2 - 3x + 2$. Write an expression that models the area of the pumpkins.

In Exercises 17–24, find the product. *(See Example 3.)*

17. $7x^3(5x^2 + 3x + 1)$

18. $-4x^5(11x^3 + 2x^2 + 9x + 1)$

19. $(5x^2 - 4x + 6)(-2x + 3)$

20. $(-x - 3)(2x^2 + 5x + 8)$

21. $(x^2 - 2x - 4)(x^2 - 3x - 5)$

22. $(3x^2 + x - 2)(-4x^2 - 2x - 1)$

23. $(3x^3 - 9x + 7)(x^2 - 2x + 1)$

24. $(4x^2 - 8x - 2)(x^4 + 3x^2 + 4x)$

ERROR ANALYSIS In Exercises 25 and 26, describe and correct the error in performing the operation.

25. ✗

$$\begin{aligned}(x^2 - 3x + 4) - (x^3 + 7x - 2) &\\ = x^2 - 3x + 4 - x^3 + 7x - 2 &\\ = -x^3 + x^2 + 4x + 2 &\end{aligned}$$

26. ✗

$$\begin{aligned}(2x - 7)^3 &= (2x)^3 - 7^3 \\ &= 8x^3 - 343\end{aligned}$$

In Exercises 27–32, find the product of the binomials. *(See Example 4.)*

27. $(x - 3)(x + 2)(x + 4)$

28. $(x - 5)(x + 2)(x - 6)$

29. $(x - 2)(3x + 1)(4x - 3)$

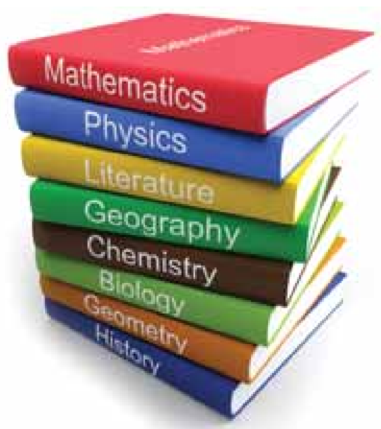
30. $(2x + 5)(x - 2)(3x + 4)$

31. $(3x - 4)(5 - 2x)(4x + 1)$

32. $(4 - 5x)(1 - 2x)(3x + 2)$

33. REASONING Prove the polynomial identity $(a + b)(a - b) = a^2 - b^2$. Then give an example of two whole numbers greater than 10 that can be multiplied using mental math and the given identity. Justify your answers. *(See Example 5.)*

34. NUMBER SENSE You have been asked to order textbooks for your class. You need to order 29 textbooks that cost \$31 each. Explain how you can use the polynomial identity $(a + b)(a - b) = a^2 - b^2$ and mental math to find the total cost of the textbooks.



In Exercises 35–42, find the product. *(See Example 6.)*

35. $(x - 9)(x + 9)$

36. $(m + 6)^2$

37. $(3c - 5)^2$

38. $(2y - 5)(2y + 5)$

39. $(7h + 4)^2$

40. $(9g - 4)^2$

41. $(2k + 6)^3$

42. $(4n - 3)^3$

In Exercises 43–48, use Pascal's Triangle to expand the binomial. *(See Example 7.)*

43. $(2t + 4)^3$

44. $(6m + 2)^2$

45. $(2q - 3)^4$

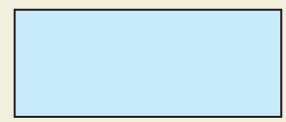
46. $(g + 2)^5$

47. $(yz + 1)^5$

48. $(np - 1)^4$

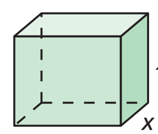
49. COMPARING METHODS Find the product of the expression $(a^2 + 4b^2)^2(3a^2 - b^2)^2$ using two different methods. Which method do you prefer? Explain.

50. THOUGHT PROVOKING Adjoin one or more polygons to the rectangle to form a single new polygon whose perimeter is double that of the rectangle. Find the perimeter of the new polygon.



MATHEMATICAL CONNECTIONS In Exercises 51 and 52, write an expression for the volume of the figure as a polynomial in standard form.

51. $V = \ell wh$



52. $V = \pi r^2 h$

53. MODELING WITH MATHEMATICS Two people make three deposits into their bank accounts earning the same simple interest rate r.

Person A — Account No. 2-5384100608

Date	Transaction	Amount
01/01/2012	Deposit	\$2000.00
01/01/2013	Deposit	\$3000.00
01/01/2014	Deposit	\$1000.00

Person B — Account No. 1-5233032905

Date	Transaction	Amount
01/01/2012	Deposit	\$5000.00
01/01/2013	Deposit	\$1000.00
01/01/2014	Deposit	\$4000.00

Person A's account is worth

$$2000(1 + r)^3 + 3000(1 + r)^2 + 1000(1 + r)$$

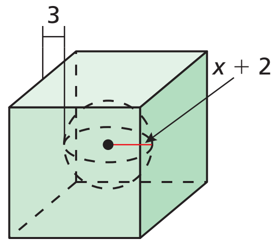
on January 1, 2015.

a. Write a polynomial for the value of Person B's account on January 1, 2015.

b. Write the total value of the two accounts as a polynomial in standard form. Then interpret the coefficients of the polynomial.

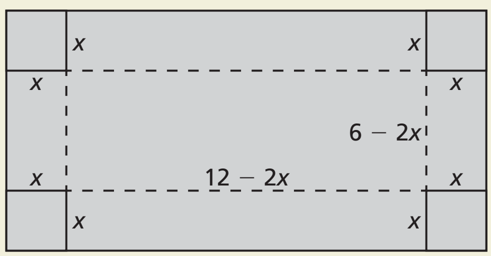
c. Suppose their interest rate is 0.05. What is the total value of the two accounts on January 1, 2015?

54. Find an expression for the volume of the cube outside the sphere.



55. MAKING AN ARGUMENT Your friend claims the sum of two binomials is always a binomial and the product of two binomials is always a trinomial. Is your friend correct? Explain your reasoning.

56. HOW DO YOU SEE IT? You make a tin box by cutting x-inch-by-x-inch pieces of tin off the corners of a rectangle and folding up each side. The plan for your box is shown.



a. What are the dimensions of the original piece of tin?

b. Write a function that represents the volume of the box. Without multiplying, determine its degree.

USING TOOLS **In Exercises 57–60, use a graphing calculator to make a conjecture about whether the two functions are equivalent. Explain your reasoning.**

57. $f(x) = (2x - 3)^3$; $g(x) = 8x^3 - 36x^2 + 54x - 27$

58. $h(x) = (x + 2)^5$;
$k(x) = x^5 + 10x^4 + 40x^3 + 80x^2 + 64x$

59. $f(x) = (-x - 3)^4$;
$g(x) = x^4 + 12x^3 + 54x^2 + 108x + 80$

60. $f(x) = (-x + 5)^3$; $g(x) = -x^3 + 15x^2 - 75x + 125$

61. REASONING Copy Pascal's Triangle and add rows for $n = 6, 7, 8, 9$, and 10. Use the new rows to expand $(x + 3)^7$ and $(x - 5)^9$.

62. ABSTRACT REASONING You are given the function $f(x) = (x + a)(x + b)(x + c)(x + d)$. When $f(x)$ is written in standard form, show that the coefficient of x^3 is the sum of a, b, c, and d, and the constant term is the product of a, b, c, and d.

63. DRAWING CONCLUSIONS Let $g(x) = 12x^4 + 8x + 9$ and $h(x) = 3x^5 + 2x^3 - 7x + 4$.

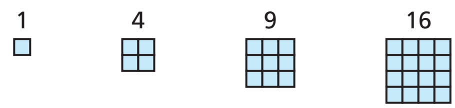
a. What is the degree of the polynomial $g(x) + h(x)$?

b. What is the degree of the polynomial $g(x) - h(x)$?

c. What is the degree of the polynomial $g(x) \cdot h(x)$?

d. In general, if $g(x)$ and $h(x)$ are polynomials such that $g(x)$ has degree m and $h(x)$ has degree n, and $m > n$, what are the degrees of $g(x) + h(x)$, $g(x) - h(x)$, and $g(x) \cdot h(x)$?

64. FINDING A PATTERN In this exercise, you will explore the sequence of square numbers. The first four square numbers are represented below.



a. Find the differences between consecutive square numbers. Explain what you notice.

b. Show how the polynomial identity $(n + 1)^2 - n^2 = 2n + 1$ models the differences between square numbers.

c. Prove the polynomial identity in part (b).

65. CRITICAL THINKING Recall that a Pythagorean triple is a set of positive integers a, b, and c such that $a^2 + b^2 = c^2$. The numbers 3, 4, and 5 form a Pythagorean triple because $3^2 + 4^2 = 5^2$. You can use the polynomial identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ to generate other Pythagorean triples.

a. Prove the polynomial identity is true by showing that the simplified expressions for the left and right sides are the same.

b. Use the identity to generate the Pythagorean triple when $x = 6$ and $y = 5$.

c. Verify that your answer in part (b) satisfies $a^2 + b^2 = c^2$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the operation. Write the answer in standard form. *(Section 3.2)*

66. $(3 - 2i) + (5 + 9i)$

67. $(12 + 3i) - (7 - 8i)$

68. $(7i)(-3i)$

69. $(4 + i)(2 - i)$