4.4 Factoring Polynomials

Essential Question

How can you factor a polynomial?

COMMON CORE

Learning Standards
HSA-SSE.A.2
HSA-APR.B.2
HSA-APR.B.3

EXPLORATION 1 Factoring Polynomials

Work with a partner. Match each polynomial equation with the graph of its related polynomial function. Use the x-intercepts of the graph to write each polynomial in factored form. Explain your reasoning.

a. $x^2 + 5x + 4 = 0$

b. $x^3 - 2x^2 - x + 2 = 0$

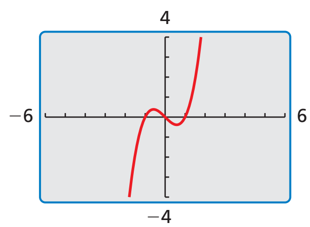
c. $x^3 + x^2 - 2x = 0$

d. $x^3 - x = 0$

e. $x^4 - 5x^2 + 4 = 0$

f. $x^4 - 2x^3 - x^2 + 2x = 0$

A.



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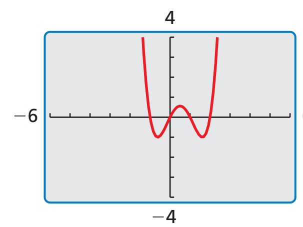

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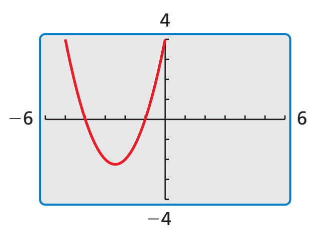

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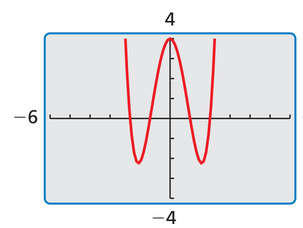

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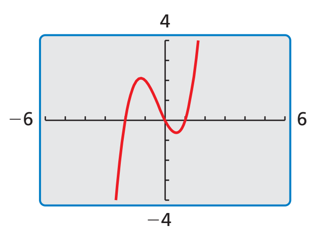

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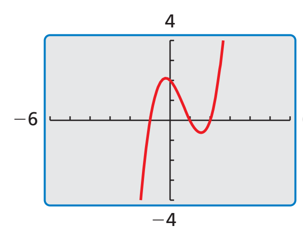

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to check your answers to problems and continually ask yourself, "Does this make sense?"

EXPLORATION 2 Factoring Polynomials

Work with a partner. Use the x-intercepts of the graph of the polynomial function to write each polynomial in factored form. Explain your reasoning. Check your answers by multiplying.

a. $f(x) = x^2 - x - 2$

b. $f(x) = x^3 - x^2 - 2x$

c. $f(x) = x^3 - 2x^2 - 3x$

d. $f(x) = x^3 - 3x^2 - x + 3$

e. $f(x) = x^4 + 2x^3 - x^2 - 2x$

f. $f(x) = x^4 - 10x^2 + 9$

Communicate Your Answer

3. How can you factor a polynomial?

4. What information can you obtain about the graph of a polynomial function written in factored form?

4.4 Lesson

Core Vocabulary

factored completely, *p. 180*
factor by grouping, *p. 181*
quadratic form, *p. 181*

Previous
zero of a function
synthetic division

What You Will Learn

- Factor polynomials.
- Use the Factor Theorem.

Factoring Polynomials

Previously, you factored quadratic polynomials. You can also factor polynomials with degree greater than 2. Some of these polynomials can be *factored completely* using techniques you have previously learned. A factorable polynomial with integer coefficients is **factored completely** when it is written as a product of unfactorable polynomials with integer coefficients.

EXAMPLE 1 Finding a Common Monomial Factor

Factor each polynomial completely.

a. $x^3 - 4x^2 - 5x$ **b.** $3y^5 - 48y^3$ **c.** $5z^4 + 30z^3 + 45z^2$

SOLUTION

a. $x^3 - 4x^2 - 5x = x(x^2 - 4x - 5)$ Factor common monomial.

$= x(x - 5)(x + 1)$ Factor trinomial.

b. $3y^5 - 48y^3 = 3y^3(y^2 - 16)$ Factor common monomial.

$= 3y^3(y - 4)(y + 4)$ Difference of Two Squares Pattern

c. $5z^4 + 30z^3 + 45z^2 = 5z^2(z^2 + 6z + 9)$ Factor common monomial.

$= 5z^2(z + 3)^2$ Perfect Square Trinomial Pattern

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Factor the polynomial completely.

1. $x^3 - 7x^2 + 10x$ **2.** $3n^7 - 75n^5$ **3.** $8m^5 - 16m^4 + 8m^3$

In part (b) of Example 1, the special factoring pattern for the difference of two squares was used to factor the expression completely. There are also factoring patterns that you can use to factor the sum or difference of two *cubes*.

Core Concept

Special Factoring Patterns

Sum of Two Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example

$64x^3 + 1 = (4x)^3 + 1^3$

$= (4x + 1)(16x^2 - 4x + 1)$

Difference of Two Cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example

$27x^3 - 8 = (3x)^3 - 2^3$

$= (3x - 2)(9x^2 + 6x + 4)$

EXAMPLE 2 Factoring the Sum or Difference of Two Cubes

Factor (a) $x^3 - 125$ and (b) $16s^5 + 54s^2$ completely.

SOLUTION

a. $x^3 - 125 = x^3 - 5^3$ — Write as $a^3 - b^3$.

$= (x - 5)(x^2 + 5x + 25)$ — Difference of Two Cubes Pattern

b. $16s^5 + 54s^2 = 2s^2(8s^3 + 27)$ — Factor common monomial.

$= 2s^2\,[(2s)^3 + 3^3]$ — Write $8s^3 + 27$ as $a^3 + b^3$.

$= 2s^2(2s + 3)(4s^2 - 6s + 9)$ — Sum of Two Cubes Pattern

For some polynomials, you can **factor by grouping** pairs of terms that have a common monomial factor. The pattern for factoring by grouping is shown below.

$$ra + rb + sa + sb = r(a + b) + s(a + b)$$
$$= (r + s)(a + b)$$

EXAMPLE 3 Factoring by Grouping

Factor $z^3 + 5z^2 - 4z - 20$ completely.

SOLUTION

$z^3 + 5z^2 - 4z - 20 = z^2(z + 5) - 4(z + 5)$ — Factor by grouping.

$= (z^2 - 4)(z + 5)$ — Distributive Property

$= (z - 2)(z + 2)(z + 5)$ — Difference of Two Squares Pattern

An expression of the form $au^2 + bu + c$, where u is an algebraic expression, is said to be in **quadratic form**. The factoring techniques you have studied can sometimes be used to factor such expressions.

LOOKING FOR STRUCTURE

The expression $16x^4 - 81$ is in quadratic form because it can be written as $u^2 - 81$ where $u = 4x^2$.

EXAMPLE 4 Factoring Polynomials in Quadratic Form

Factor (a) $16x^4 - 81$ and (b) $3p^8 + 15p^5 + 18p^2$ completely.

SOLUTION

a. $16x^4 - 81 = (4x^2)^2 - 9^2$ — Write as $a^2 - b^2$.

$= (4x^2 + 9)(4x^2 - 9)$ — Difference of Two Squares Pattern

$= (4x^2 + 9)(2x - 3)(2x + 3)$ — Difference of Two Squares Pattern

b. $3p^8 + 15p^5 + 18p^2 = 3p^2(p^6 + 5p^3 + 6)$ — Factor common monomial.

$= 3p^2(p^3 + 3)(p^3 + 2)$ — Factor trinomial in quadratic form.

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Factor the polynomial completely.

4. $a^3 + 27$

5. $6z^5 - 750z^2$

6. $x^3 + 4x^2 - x - 4$

7. $3y^3 + y^2 + 9y + 3$

8. $-16n^4 + 625$

9. $5w^6 - 25w^4 + 30w^2$

The Factor Theorem

When dividing polynomials in the previous section, the examples had nonzero remainders. Suppose the remainder is 0 when a polynomial $f(x)$ is divided by $x - k$. Then,

$$\frac{f(x)}{x-k} = q(x) + \frac{0}{x-k} = q(x)$$

where $q(x)$ is the quotient polynomial. Therefore, $f(x) = (x - k) \cdot q(x)$, so that $x - k$ is a factor of $f(x)$. This result is summarized by the *Factor Theorem*, which is a special case of the Remainder Theorem.

READING

In other words, $x - k$ is a factor of $f(x)$ if and only if k is a zero of f.

Core Concept

The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

STUDY TIP

In part (b), notice that direct substitution would have resulted in more difficult computations than synthetic division.

EXAMPLE 5 Determining Whether a Linear Binomial Is a Factor

Determine whether (a) $x - 2$ is a factor of $f(x) = x^2 + 2x - 4$ and (b) $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$.

SOLUTION

a. Find $f(2)$ by direct substitution.

$$\begin{aligned} f(2) &= 2^2 + 2(2) - 4 \\ &= 4 + 4 - 4 \\ &= 4 \end{aligned}$$

Because $f(2) \neq 0$, the binomial $x - 2$ is not a factor of $f(x) = x^2 + 2x - 4$.

b. Find $f(-5)$ by synthetic division.

$$\begin{array}{r|rrrrr} -5 & 3 & 15 & -1 & 0 & 25 \\ & & -15 & 0 & 5 & -25 \\ \hline & 3 & 0 & -1 & 5 & 0 \end{array}$$

Because $f(-5) = 0$, the binomial $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$.

EXAMPLE 6 Factoring a Polynomial

Show that $x + 3$ is a factor of $f(x) = x^4 + 3x^3 - x - 3$. Then factor $f(x)$ completely.

SOLUTION

Show that $f(-3) = 0$ by synthetic division.

$$\begin{array}{r|rrrrr} -3 & 1 & 3 & 0 & -1 & -3 \\ & & -3 & 0 & 0 & 3 \\ \hline & 1 & 0 & 0 & -1 & 0 \end{array}$$

Because $f(-3) = 0$, you can conclude that $x + 3$ is a factor of $f(x)$ by the Factor Theorem. Use the result to write $f(x)$ as a product of two factors and then factor completely.

$f(x) = x^4 + 3x^3 - x - 3$ — Write original polynomial.

$= (x + 3)(x^3 - 1)$ — Write as a product of two factors.

$= (x + 3)(x - 1)(x^2 + x + 1)$ — Difference of Two Cubes Pattern

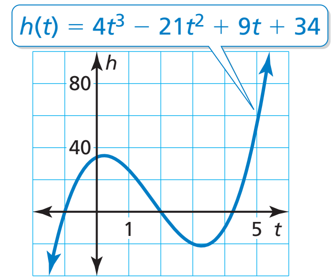
ANOTHER WAY

Notice that you can factor $f(x)$ by grouping.

$$\begin{aligned} f(x) &= x^3(x + 3) - 1(x + 3) \\ &= (x^3 - 1)(x + 3) \\ &= (x + 3)(x - 1) \cdot (x^2 + x + 1) \end{aligned}$$

Because the x-intercepts of the graph of a function are the zeros of the function, you can use the graph to approximate the zeros. You can check the approximations using the Factor Theorem.

EXAMPLE 7 Real-Life Application



During the first 5 seconds of a roller coaster ride, the function $h(t) = 4t^3 - 21t^2 + 9t + 34$ represents the height h (in feet) of the roller coaster after t seconds. How long is the roller coaster at or below ground level in the first 5 seconds?

SOLUTION

1. **Understand the Problem** You are given a function rule that represents the height of a roller coaster. You are asked to determine how long the roller coaster is at or below ground during the first 5 seconds of the ride.
2. **Make a Plan** Use a graph to estimate the zeros of the function and check using the Factor Theorem. Then use the zeros to describe where the graph lies below the t-axis.
3. **Solve the Problem** From the graph, two of the zeros appear to be -1 and 2. The third zero is between 4 and 5.

Step 1 Determine whether -1 is a zero using synthetic division.

-1	4	-21	9	34
		-4	25	-34
	4	-25	34	0

$h(-1) = 0$, so -1 is a zero of h and $t + 1$ is a factor of $h(t)$.

Step 2 Determine whether 2 is a zero. If 2 is also a zero, then $t - 2$ is a factor of the resulting quotient polynomial. Check using synthetic division.

2	4	-25	34
		8	-34
	4	-17	0

The remainder is 0, so $t - 2$ is a factor of $h(t)$ and 2 is a zero of h.

STUDY TIP

You could also check that 2 is a zero using the original function, but using the quotient polynomial helps you find the remaining factor.

So, $h(t) = (t + 1)(t - 2)(4t - 17)$. The factor $4t - 17$ indicates that the zero between 4 and 5 is $\frac{17}{4}$, or 4.25.

▶ The zeros are -1, 2, and 4.25. Only $t = 2$ and $t = 4.25$ occur in the first 5 seconds. The graph shows that the roller coaster is at or below ground level for $4.25 - 2 = 2.25$ seconds.

4. **Look Back** Use a table of values to verify the positive zeros and heights between the zeros.

X	Y1	
.5	33.75	
1.25	20.25	
2	0	zero
2.75	−16.88	negative
3.5	−20.25	negative
4.25	0	zero
5	54	
X=2		

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10. Determine whether $x - 4$ is a factor of $f(x) = 2x^2 + 5x - 12$.
11. Show that $x - 6$ is a factor of $f(x) = x^3 - 5x^2 - 6x$. Then factor $f(x)$ completely.
12. In Example 7, does your answer change when you first determine whether 2 is a zero and then whether -1 is a zero? Justify your answer.

4.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The expression $9x^4 - 49$ is in _________ form because it can be written as $u^2 - 49$ where $u =$ _____.

2. **VOCABULARY** Explain when you should try factoring a polynomial by grouping.

3. **WRITING** How do you know when a polynomial is factored completely?

4. **WRITING** Explain the Factor Theorem and why it is useful.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, factor the polynomial completely. *(See Example 1.)*

5. $x^3 - 2x^2 - 24x$

6. $4k^5 - 100k^3$

7. $3p^5 - 192p^3$

8. $2m^6 - 24m^5 + 64m^4$

9. $2q^4 + 9q^3 - 18q^2$

10. $3r^6 - 11r^5 - 20r^4$

11. $10w^{10} - 19w^9 + 6w^8$

12. $18v^9 + 33v^8 + 14v^7$

In Exercises 13–20, factor the polynomial completely. *(See Example 2.)*

13. $x^3 + 64$

14. $y^3 + 512$

15. $g^3 - 343$

16. $c^3 - 27$

17. $3h^9 - 192h^6$

18. $9n^6 - 6561n^3$

19. $16t^7 + 250t^4$

20. $135z^{11} - 1080z^8$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in factoring the polynomial.

21. ✗
$$3x^3 + 27x = 3x(x^2 + 9)$$
$$= 3x(x + 3)(x - 3)$$

22. ✗
$$x^9 + 8x^3 = (x^3)^3 + (2x)^3$$
$$= (x^3 + 2x)[(x^3)^2 - (x^3)(2x) + (2x)^2]$$
$$= (x^3 + 2x)(x^6 - 2x^4 + 4x^2)$$

In Exercises 23–30, factor the polynomial completely. *(See Example 3.)*

23. $y^3 - 5y^2 + 6y - 30$

24. $m^3 - m^2 + 7m - 7$

25. $3a^3 + 18a^2 + 8a + 48$

26. $2k^3 - 20k^2 + 5k - 50$

27. $x^3 - 8x^2 - 4x + 32$

28. $z^3 - 5z^2 - 9z + 45$

29. $4q^3 - 16q^2 - 9q + 36$

30. $16n^3 + 32n^2 - n - 2$

In Exercises 31–38, factor the polynomial completely. *(See Example 4.)*

31. $49k^4 - 9$

32. $4m^4 - 25$

33. $c^4 + 9c^2 + 20$

34. $y^4 - 3y^2 - 28$

35. $16z^4 - 81$

36. $81a^4 - 256$

37. $3r^8 + 3r^5 - 60r^2$

38. $4n^{12} - 32n^7 + 48n^2$

In Exercises 39–44, determine whether the binomial is a factor of the polynomial function. *(See Example 5.)*

39. $f(x) = 2x^3 + 5x^2 - 37x - 60;\ x - 4$

40. $g(x) = 3x^3 - 28x^2 + 29x + 140;\ x + 7$

41. $h(x) = 6x^5 - 15x^4 - 9x^3;\ x + 3$

42. $g(x) = 8x^5 - 58x^4 + 60x^3 + 140;\ x - 6$

43. $h(x) = 6x^4 - 6x^3 - 84x^2 + 144x;\ x + 4$

44. $t(x) = 48x^4 + 36x^3 - 138x^2 - 36x;\ x + 2$

In Exercises 45–50, show that the binomial is a factor of the polynomial. Then factor the function completely. *(See Example 6.)*

45. $g(x) = x^3 - x^2 - 20x;\ x + 4$

46. $t(x) = x^3 - 5x^2 - 9x + 45;\ x - 5$

47. $f(x) = x^4 - 6x^3 - 8x + 48;\ x - 6$

48. $s(x) = x^4 + 4x^3 - 64x - 256;\ x + 4$

49. $r(x) = x^3 - 37x + 84;\ x + 7$

50. $h(x) = x^3 - x^2 - 24x - 36;\ x + 2$

ANALYZING RELATIONSHIPS In Exercises 51–54, match the function with the correct graph. Explain your reasoning.

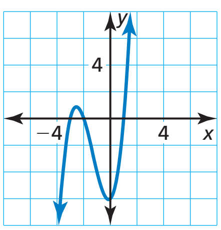
51. $f(x) = (x - 2)(x - 3)(x + 1)$

52. $g(x) = x(x + 2)(x + 1)(x - 2)$

53. $h(x) = (x + 2)(x + 3)(x - 1)$

54. $k(x) = x(x - 2)(x - 1)(x + 2)$

A.



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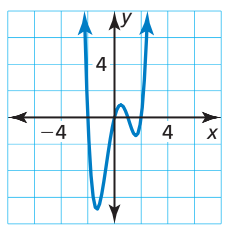

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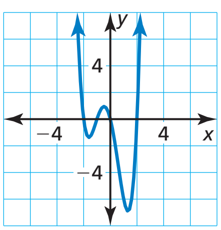

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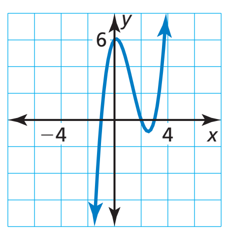
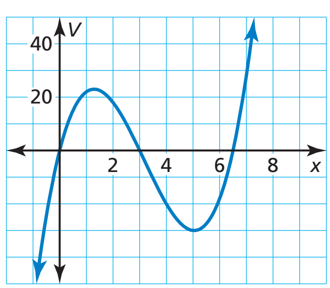
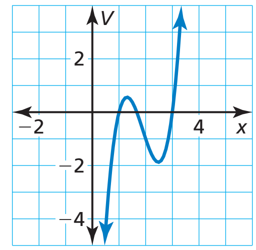


55. MODELING WITH MATHEMATICS The volume (in cubic inches) of a shipping box is modeled by $V = 2x^3 - 19x^2 + 39x$, where x is the length (in inches). Determine the values of x for which the model makes sense. Explain your reasoning. *(See Example 6.)*



56. MODELING WITH MATHEMATICS The volume (in cubic inches) of a rectangular birdcage can be modeled by $V = 3x^3 - 17x^2 + 29x - 15$, where x is the length (in inches). Determine the values of x for which the model makes sense. Explain your reasoning.



USING STRUCTURE In Exercises 57–64, use the method of your choice to factor the polynomial completely. Explain your reasoning.

57. $a^6 + a^5 - 30a^4$

58. $8m^3 - 343$

59. $z^3 - 7z^2 - 9z + 63$

60. $2p^8 - 12p^5 + 16p^2$

61. $64r^3 + 729$

62. $5x^5 - 10x^4 - 40x^3$

63. $16n^4 - 1$

64. $9k^3 - 24k^2 + 3k - 8$

65. REASONING Determine whether each polynomial is factored completely. If not, factor completely.

a. $7z^4(2z^2 - z - 6)$

b. $(2 - n)(n^2 + 6n)(3n - 11)$

c. $3(4y - 5)(9y^2 - 6y - 4)$

66. PROBLEM SOLVING The profit P (in millions of dollars) for a T-shirt manufacturer can be modeled by $P = -x^3 + 4x^2 + x$, where x is the number (in millions) of T-shirts produced. Currently the company produces 4 million T-shirts and makes a profit of \$4 million. What lesser number of T-shirts could the company produce and still make the same profit?

67. PROBLEM SOLVING The profit P (in millions of dollars) for a shoe manufacturer can be modeled by $P = -21x^3 + 46x$, where x is the number (in millions) of shoes produced. The company now produces 1 million shoes and makes a profit of \$25 million, but it would like to cut back production. What lesser number of shoes could the company produce and still make the same profit?

68. THOUGHT PROVOKING Fill in the blank of the divisor so that the remainder is 0. Justify your answer.

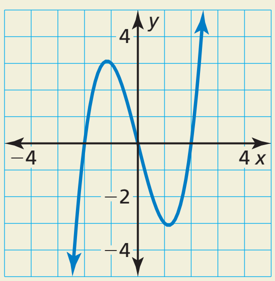
$f(x) = x^3 - 3x^2 - 4x$; $(x + \_\_\_)$

69. COMPARING METHODS You are taking a test where calculators are not permitted. One question asks you to evaluate $g(7)$ for the function $g(x) = x^3 - 7x^2 - 4x + 28$. You use the Factor Theorem and synthetic division and your friend uses direct substitution. Whose method do you prefer? Explain your reasoning.

70. MAKING AN ARGUMENT You divide $f(x)$ by $(x - a)$ and find that the remainder does not equal 0. Your friend concludes that $f(x)$ cannot be factored. Is your friend correct? Explain your reasoning.

71. CRITICAL THINKING What is the value of k such that $x - 7$ is a factor of $h(x) = 2x^3 - 13x^2 - kx + 105$? Justify your answer.

72. HOW DO YOU SEE IT? Use the graph to write an equation of the cubic function in factored form. Explain your reasoning.



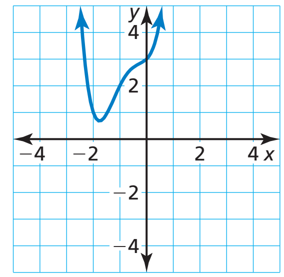
73. ABSTRACT REASONING Factor each polynomial completely.

a. $7ac^2 + bc^2 - 7ad^2 - bd^2$

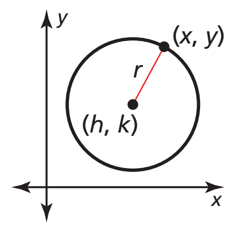
b. $x^{2n} - 2x^n + 1$

c. $a^5b^2 - a^2b^4 + 2a^4b - 2ab^3 + a^3 - b^2$

74. REASONING The graph of the function $f(x) = x^4 + 3x^3 + 2x^2 + x + 3$ is shown. Can you use the Factor Theorem to factor $f(x)$? Explain.



75. MATHEMATICAL CONNECTIONS The standard equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$. Rewrite each equation of a circle in standard form. Identify the center and radius of the circle. Then graph the circle.



a. $x^2 + 6x + 9 + y^2 = 25$

b. $x^2 - 4x + 4 + y^2 = 9$

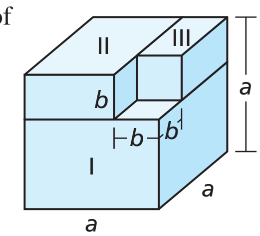
c. $x^2 - 8x + 16 + y^2 + 2y + 1 = 36$

76. CRITICAL THINKING Use the diagram to complete parts (a)–(c).

a. Explain why $a^3 - b^3$ is equal to the sum of the volumes of the solids I, II, and III.

b. Write an algebraic expression for the volume of each of the three solids. Leave your expressions in factored form.

c. Use the results from part (a) and part (b) to derive the factoring pattern $a^3 - b^3$.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the quadratic equation by factoring. *(Section 3.1)*

77. $x^2 - x - 30 = 0$

78. $2x^2 - 10x - 72 = 0$

79. $3x^2 - 11x + 10 = 0$

80. $9x^2 - 28x + 3 = 0$

Solve the quadratic equation by completing the square. *(Section 3.3)*

81. $x^2 - 12x + 36 = 144$

82. $x^2 - 8x - 11 = 0$

83. $3x^2 + 30x + 63 = 0$

84. $4x^2 + 36x - 4 = 0$