6.1 Exponential Growth and Decay Functions

COMMON CORE

Learning Standards
HSA-SSE.B.3c
HSF-IF.C.7e
HSF-IF.C.8b
HSF-LE.A.2
HSF-LE.B.5

Essential Question

What are some of the characteristics of the graph of an exponential function?

You can use a graphing calculator to evaluate an exponential function. For example, consider the exponential function $f(x) = 2^x$.

Function Value	Graphing Calculator Keystrokes	Display
$f(-3.1) = 2^{-3.1}$	2 ^ (–) 3.1 ENTER	0.1166291
$f\left(\frac{2}{3}\right) = 2^{2/3}$	2 ^ (2 ÷ 3) ENTER	1.5874011

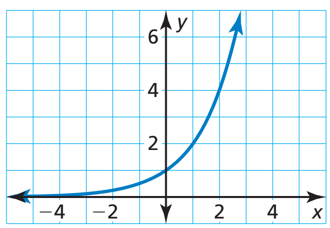
EXPLORATION 1 Identifying Graphs of Exponential Functions

Work with a partner. Match each exponential function with its graph. Use a table of values to sketch the graph of the function, if necessary.

a. $f(x) = 2^x$ **b.** $f(x) = 3^x$ **c.** $f(x) = 4^x$

d. $f(x) = \left(\frac{1}{2}\right)^x$ **e.** $f(x) = \left(\frac{1}{3}\right)^x$ **f.** $f(x) = \left(\frac{1}{4}\right)^x$

A.



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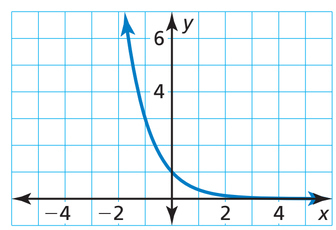

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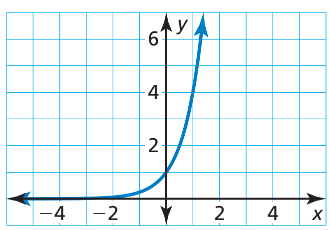

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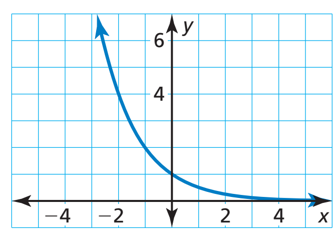

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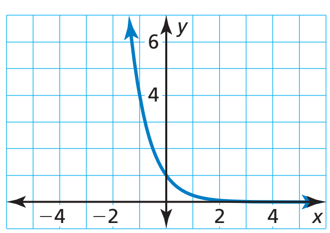

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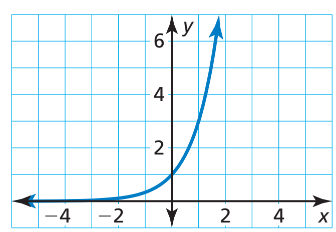


EXPLORATION 2 Characteristics of Graphs of Exponential Functions

Work with a partner. Use the graphs in Exploration 1 to determine the domain, range, and y-intercept of the graph of $f(x) = b^x$, where b is a positive real number other than 1. Explain your reasoning.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

Communicate Your Answer

3. What are some of the characteristics of the graph of an exponential function?

4. In Exploration 2, is it possible for the graph of $f(x) = b^x$ to have an x-intercept? Explain your reasoning.

6.1 Lesson

Core Vocabulary

exponential function, *p. 296*
exponential growth function, *p. 296*
growth factor, *p. 296*
asymptote, *p. 296*
exponential decay function, *p. 296*
decay factor, *p. 296*

Previous
properties of exponents

What You Will Learn

- Graph exponential growth and decay functions.
- Use exponential models to solve real-life problems.

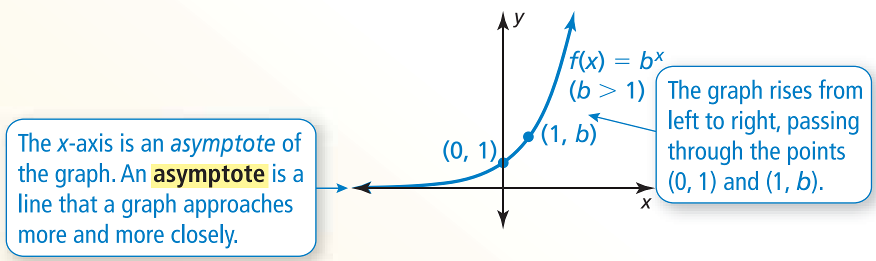
Exponential Growth and Decay Functions

An **exponential function** has the form $y = ab^x$, where $a \neq 0$ and the base b is a positive real number other than 1. If $a > 0$ and $b > 1$, then $y = ab^x$ is an **exponential growth function**, and b is called the **growth factor**. The simplest type of exponential growth function has the form $y = b^x$.

Core Concept

Parent Function for Exponential Growth Functions

The function $f(x) = b^x$, where $b > 1$, is the parent function for the family of exponential growth functions with base b. The graph shows the general shape of an exponential growth function.



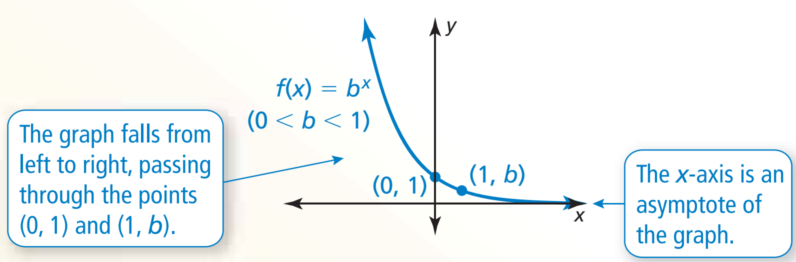
The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

If $a > 0$ and $0 < b < 1$, then $y = ab^x$ is an **exponential decay function**, and b is called the **decay factor**.

Core Concept

Parent Function for Exponential Decay Functions

The function $f(x) = b^x$, where $0 < b < 1$, is the parent function for the family of exponential decay functions with base b. The graph shows the general shape of an exponential decay function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

![EXAMPLE 1] **Graphing Exponential Growth and Decay Functions**

Tell whether each function represents *exponential growth* or *exponential decay*. Then graph the function.

a. $y = 2^x$ **b.** $y = \left(\frac{1}{2}\right)^x$

SOLUTION

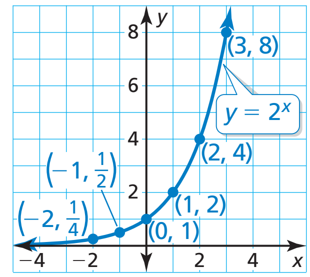
a. Step 1 Identify the value of the base. The base, 2, is greater than 1, so the function represents exponential growth.

Step 2 Make a table of values.

x	−2	−1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Step 3 Plot the points from the table.

Step 4 Draw, from *left to right*, a smooth curve that begins just above the *x*-axis, passes through the plotted points, and moves up to the right.



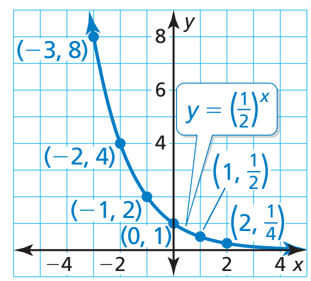
b. Step 1 Identify the value of the base. The base, $\frac{1}{2}$, is greater than 0 and less than 1, so the function represents exponential decay.

Step 2 Make a table of values.

x	−3	−2	−1	0	1	2
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Step 3 Plot the points from the table.

Step 4 Draw, from *right to left*, a smooth curve that begins just above the *x*-axis, passes through the plotted points, and moves up to the left.



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Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

1. $y = 4^x$

2. $y = \left(\frac{2}{3}\right)^x$

3. $f(x) = (0.25)^x$

4. $f(x) = (1.5)^x$

Exponential Models

Some real-life quantities increase or decrease by a fixed percent each year (or some other time period). The amount y of such a quantity after t years can be modeled by one of these equations.

Exponential Growth Model $y = a(1 + r)^t$

Exponential Decay Model $y = a(1 - r)^t$

Note that a is the initial amount and r is the percent increase or decrease written as a decimal. The quantity $1 + r$ is the growth factor, and $1 - r$ is the decay factor.

EXAMPLE 2 Solving a Real-Life Problem

The value of a car y (in thousands of dollars) can be approximated by the model $y = 25(0.85)^t$, where t is the number of years since the car was new.

a. Tell whether the model represents exponential growth or exponential decay.

b. Identify the annual percent increase or decrease in the value of the car.

c. Estimate when the value of the car will be \$8000.

REASONING QUANTITATIVELY

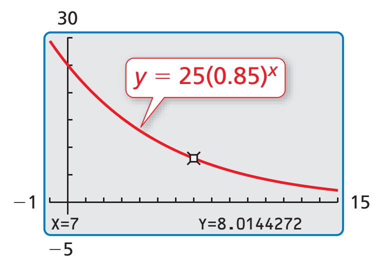
The percent decrease, 15%, tells you how much value the car *loses* each year. The decay factor, 0.85, tells you what fraction of the car's value *remains* each year.

SOLUTION

a. The base, 0.85, is greater than 0 and less than 1, so the model represents exponential decay.

b. Because t is given in years and the decay factor $0.85 = 1 - 0.15$, the annual percent decrease is 0.15, or 15%.

c. Use the *trace* feature of a graphing calculator to determine that $y \approx 8$ when $t = 7$. After 7 years, the value of the car will be about \$8000.



EXAMPLE 3 Writing an Exponential Model

In 2000, the world population was about 6.09 billion. During the next 13 years, the world population increased by about 1.18% each year.

a. Write an exponential growth model giving the population y (in billions) t years after 2000. Estimate the world population in 2005.

b. Estimate the year when the world population was 7 billion.

SOLUTION

a. The initial amount is $a = 6.09$, and the percent increase is $r = 0.0118$. So, the exponential growth model is

$$y = a(1 + r)^t \quad \text{Write exponential growth model.}$$

$$= 6.09(1 + 0.0118)^t \quad \text{Substitute 6.09 for } a \text{ and 0.0118 for } r.$$

$$= 6.09(1.0118)^t. \quad \text{Simplify.}$$

Using this model, you can estimate the world population in 2005 ($t = 5$) to be $y = 6.09(1.0118)^5 \approx 6.46$ billion.

b. Use the *table* feature of a graphing calculator to determine that $y \approx 7$ when $t = 12$. So, the world population was about 7 billion in 2012.

X	Y1	
6	6.5341	
7	6.6112	
8	6.6892	
9	6.7681	
10	6.848	
11	6.9288	
12	7.0106	

X=12

Monitoring Progress

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5. WHAT IF? In Example 2, the value of the car can be approximated by the model $y = 25(0.9)^t$. Identify the annual percent decrease in the value of the car. Estimate when the value of the car will be \$8000.

6. WHAT IF? In Example 3, assume the world population increased by 1.5% each year. Write an equation to model this situation. Estimate the year when the world population was 7 billion.

EXAMPLE 4 Rewriting an Exponential Function

The amount y (in grams) of the radioactive isotope chromium-51 remaining after t days is $y = a(0.5)^{t/28}$, where a is the initial amount (in grams). What percent of the chromium-51 decays each day?

SOLUTION

$y = a(0.5)^{t/28}$	Write original function.
$= a[(0.5)^{1/28}]^t$	Power of a Power Property
$\approx a(0.9755)^t$	Evaluate power.
$= a(1 - 0.0245)^t$	Rewrite in form $y = a(1 - r)^t$.

▶ The daily decay rate is about 0.0245, or 2.45%.

Compound interest is interest paid on an initial investment, called the *principal*, and on previously earned interest. Interest earned is often expressed as an *annual* percent, but the interest is usually compounded more than once per year. So, the exponential growth model $y = a(1 + r)^t$ must be modified for compound interest problems.

Core Concept

Compound Interest

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

EXAMPLE 5 Finding the Balance in an Account

You deposit \$9000 in an account that pays 1.46% annual interest. Find the balance after 3 years when the interest is compounded quarterly.

SOLUTION

With interest compounded quarterly (4 times per year), the balance after 3 years is

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Write compound interest formula.
$= 9000\left(1 + \frac{0.0146}{4}\right)^{4 \cdot 3}$	$P = 9000$, $r = 0.0146$, $n = 4$, $t = 3$
$\approx 9402.21.$	Use a calculator.

▶ The balance at the end of 3 years is \$9402.21.

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7. The amount y (in grams) of the radioactive isotope iodine-123 remaining after t hours is $y = a(0.5)^{t/13}$, where a is the initial amount (in grams). What percent of the iodine-123 decays each hour?

8. **WHAT IF?** In Example 5, find the balance after 3 years when the interest is compounded daily.

6.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** In the exponential growth model $y = 2.4(1.5)^x$, identify the initial amount, the growth factor, and the percent increase.

2. **WHICH ONE DOESN'T BELONG?** Which characteristic of an exponential decay function does *not* belong with the other three? Explain your reasoning.

base of 0.8	decay factor of 0.8
decay rate of 20%	80% decrease

Monitoring Progress and Modeling with Mathematics

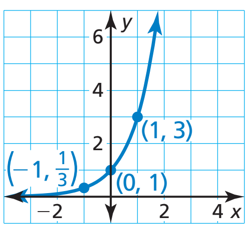
In Exercises 3–8, evaluate the expression for (a) $x = -2$ and (b) $x = 3$.

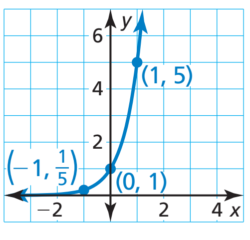
3. 2^x
4. 4^x
5. $8 \cdot 3^x$
6. $6 \cdot 2^x$
7. $5 + 3^x$
8. $2^x - 2$

In Exercises 9–18, tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(See Example 1.)*

9. $y = 6^x$
10. $y = 7^x$
11. $y = \left(\frac{1}{6}\right)^x$
12. $y = \left(\frac{1}{8}\right)^x$
13. $y = \left(\frac{4}{3}\right)^x$
14. $y = \left(\frac{2}{5}\right)^x$
15. $y = (1.2)^x$
16. $y = (0.75)^x$
17. $y = (0.6)^x$
18. $y = (1.8)^x$

ANALYZING RELATIONSHIPS **In Exercises 19 and 20, use the graph of $f(x) = b^x$ to identify the value of the base b.**

19. 

20. 

21. **MODELING WITH MATHEMATICS** The value of a mountain bike y (in dollars) can be approximated by the model $y = 200(0.75)^t$, where t is the number of years since the bike was new. *(See Example 2.)*
    a. Tell whether the model represents exponential growth or exponential decay.
    b. Identify the annual percent increase or decrease in the value of the bike.
    c. Estimate when the value of the bike will be $50.

22. **MODELING WITH MATHEMATICS** The population P (in thousands) of Austin, Texas, during a recent decade can be approximated by $y = 494.29(1.03)^t$, where t is the number of years since the beginning of the decade.
    a. Tell whether the model represents exponential growth or exponential decay.
    b. Identify the annual percent increase or decrease in population.
    c. Estimate when the population was about 590,000.

23. **MODELING WITH MATHEMATICS** In 2006, there were approximately 233 million cell phone subscribers in the United States. During the next 4 years, the number of cell phone subscribers increased by about 6% each year. *(See Example 3.)*
    a. Write an exponential growth model giving the number of cell phone subscribers y (in millions) t years after 2006. Estimate the number of cell phone subscribers in 2008.
    b. Estimate the year when the number of cell phone subscribers was about 278 million.

24. **MODELING WITH MATHEMATICS** You take a 325 milligram dosage of ibuprofen. During each subsequent hour, the amount of medication in your bloodstream decreases by about 29% each hour.

a. Write an exponential decay model giving the amount y (in milligrams) of ibuprofen in your bloodstream t hours after the initial dose.

b. Estimate how long it takes for you to have 100 milligrams of ibuprofen in your bloodstream.

JUSTIFYING STEPS **In Exercises 25 and 26, justify each step in rewriting the exponential function.**

25. $y = a(3)^{t/14}$ Write original function.

$= a[(3)^{1/14}]^t$

$\approx a(1.0816)^t$

$= a(1 + 0.0816)^t$

26. $y = a(0.1)^{t/3}$ Write original function.

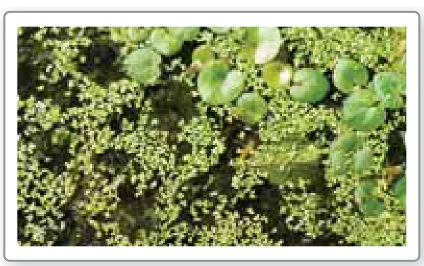
$= a[(0.1)^{1/3}]^t$

$\approx a(0.4642)^t$

$= a(1 - 0.5358)^t$

27. **PROBLEM SOLVING** When a plant or animal dies, it stops acquiring carbon-14 from the atmosphere. The amount y (in grams) of carbon-14 in the body of an organism after t years is $y = a(0.5)^{t/5730}$, where a is the initial amount (in grams). What percent of the carbon-14 is released each year? *(See Example 4.)*

28. **PROBLEM SOLVING** The number y of duckweed fronds in a pond after t days is $y = a(1230.25)^{t/16}$, where a is the initial number of fronds. By what percent does the duckweed increase each day?



In Exercises 29–36, rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. Then state the growth or decay rate.

29. $y = a(2)^{t/3}$

30. $y = a(4)^{t/6}$

31. $y = a(0.5)^{t/12}$

32. $y = a(0.25)^{t/9}$

33. $y = a\left(\frac{2}{3}\right)^{t/10}$

34. $y = a\left(\frac{5}{4}\right)^{t/22}$

35. $y = a(2)^{8t}$

36. $y = a\left(\frac{1}{3}\right)^{3t}$

37. **PROBLEM SOLVING** You deposit \$5000 in an account that pays 2.25% annual interest. Find the balance after 5 years when the interest is compounded quarterly. *(See Example 5.)*

38. **DRAWING CONCLUSIONS** You deposit \$2200 into three separate bank accounts that each pay 3% annual interest. How much interest does each account earn after 6 years?

Account	Compounding	Balance after 6 years
1	quarterly	
2	monthly	
3	daily	

39. **ERROR ANALYSIS** You invest \$500 in the stock of a company. The value of the stock decreases 2% each year. Describe and correct the error in writing a model for the value of the stock after t years.

$$y = \begin{pmatrix}\text{Initial}\\\text{amount}\end{pmatrix}\begin{pmatrix}\text{Decay}\\\text{factor}\end{pmatrix}^t$$

$$y = 500(0.02)^t$$

40. **ERROR ANALYSIS** You deposit \$250 in an account that pays 1.25% annual interest. Describe and correct the error in finding the balance after 3 years when the interest is compounded quarterly.

$$A = 250\left(1 + \frac{1.25}{4}\right)^{4 \cdot 3}$$

$$A = \$6533.29$$

In Exercises 41–44, use the given information to find the amount A in the account earning compound interest after 6 years when the principal is \$3500.

41. $r = 2.16\%$, compounded quarterly

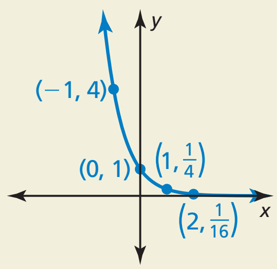
42. $r = 2.29\%$, compounded monthly

43. $r = 1.83\%$, compounded daily

44. $r = 1.26\%$, compounded monthly

45. **USING STRUCTURE** A website recorded the number y of referrals it received from social media websites over a 10-year period. The results can be modeled by $y = 2500(1.50)^t$, where t is the year and $0 \le t \le 9$. Interpret the values of a and b in this situation. What is the annual percent increase? Explain.

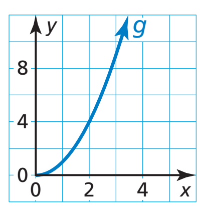
46. **HOW DO YOU SEE IT?** Consider the graph of an exponential function of the form $f(x) = ab^x$.



a. Determine whether the graph of f represents exponential growth or exponential decay.

b. What are the domain and range of the function? Explain.

47. **MAKING AN ARGUMENT** Your friend says the graph of $f(x) = 2^x$ increases at a faster rate than the graph of $g(x) = x^2$ when $x \ge 0$. Is your friend correct? Explain your reasoning.



48. **THOUGHT PROVOKING** The function $f(x) = b^x$ represents an exponential decay function. Write a second exponential decay function in terms of b and x.

49. **PROBLEM SOLVING** The population p of a small town after x years can be modeled by the function $p = 6850(1.03)^x$. What is the average rate of change in the population over the first 6 years? Justify your answer.

50. **REASONING** Consider the exponential function $f(x) = ab^x$.

a. Show that $\dfrac{f(x+1)}{f(x)} = b$.

b. Use the equation in part (a) to explain why there is no exponential function of the form $f(x) = ab^x$ whose graph passes through the points in the table below.

x	0	1	2	3	4
y	4	4	8	24	72

51. **PROBLEM SOLVING** The number E of eggs a Leghorn chicken produces per year can be modeled by the equation $E = 179.2(0.89)^{w/52}$, where w is the age (in weeks) of the chicken and $w \ge 22$.

a. Identify the decay factor and the percent decrease.

b. Graph the model.

c. Estimate the egg production of a chicken that is 2.5 years old.

d. Explain how you can rewrite the given equation so that time is measured in years rather than in weeks.

52. **CRITICAL THINKING** You buy a new stereo for \$1300 and are able to sell it 4 years later for \$275. Assume that the resale value of the stereo decays exponentially with time. Write an equation giving the resale value V (in dollars) of the stereo as a function of the time t (in years) since you bought it.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. Assume all variables are positive. *(Skills Review Handbook)*

53. $x^9 \cdot x^2$

54. $\dfrac{x^4}{x^3}$

55. $4x \cdot 6x$

56. $\left(\dfrac{4x^8}{2x^6}\right)^4$

57. $\dfrac{x + 3x}{2}$

58. $\dfrac{6x}{2} + 4x$

59. $\dfrac{12x}{4x} + 5x$

60. $(2x \cdot 3x^5)^3$