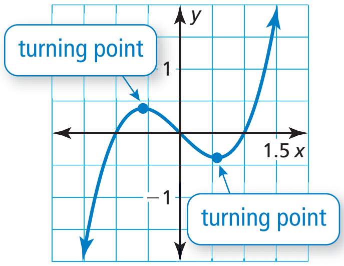
4.8 Analyzing Graphs of Polynomial Functions

Essential Question

How many turning points can the graph of a polynomial function have?

A *turning point* of the graph of a polynomial function is a point on the graph at which the function changes from

- increasing to decreasing, or
- decreasing to increasing.



EXPLORATION 1 Approximating Turning Points

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Then use a graphing calculator to approximate the coordinates of the turning points of the graph of the function. Round your answers to the nearest hundredth.

ATTENDING TO PRECISION

To be proficient in math, you need to express numerical answers with a degree of precision appropriate for the problem context.

a. $f(x) = 2x^2 + 3x - 4$

b. $f(x) = x^2 + 3x + 2$

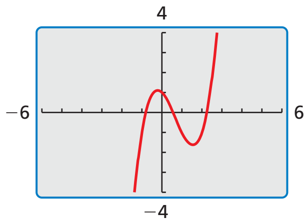
c. $f(x) = x^3 - 2x^2 - x + 1$

d. $f(x) = -x^3 + 5x - 2$

e. $f(x) = x^4 - 3x^2 + 2x - 1$

f. $f(x) = -2x^5 - x^2 + 5x + 3$

A.



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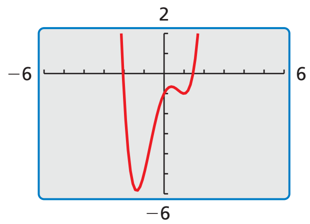

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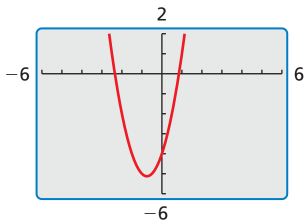

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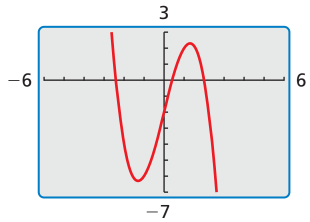

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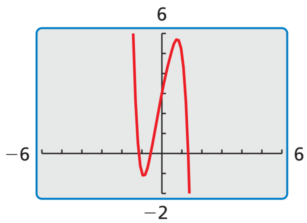

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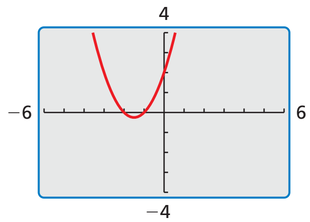


Communicate Your Answer

2. How many turning points can the graph of a polynomial function have?

3. Is it possible to sketch the graph of a cubic polynomial function that has *no* turning points? Justify your answer.

4.8 Lesson

Core Vocabulary

local maximum, *p. 214*
local minimum, *p. 214*
even function, *p. 215*
odd function, *p. 215*

Previous
end behavior
increasing
decreasing
symmetric about the *y*-axis

What You Will Learn

- Use *x*-intercepts to graph polynomial functions.
- Use the Location Principle to identify zeros of polynomial functions.
- Find turning points and identify local maximums and local minimums of graphs of polynomial functions.
- Identify even and odd functions.

Graphing Polynomial Functions

In this chapter, you have learned that zeros, factors, solutions, and *x*-intercepts are closely related concepts. Here is a summary of these relationships.

Concept Summary

Zeros, Factors, Solutions, and Intercepts

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero: k is a zero of the polynomial function f.

Factor: $x - k$ is a factor of the polynomial $f(x)$.

Solution: k is a solution (or root) of the polynomial equation $f(x) = 0$.

***x*-Intercept:** If k is a real number, then k is an x-intercept of the graph of the polynomial function f. The graph of f passes through $(k, 0)$.

EXAMPLE 1 Using *x*-Intercepts to Graph a Polynomial Function

Graph the function

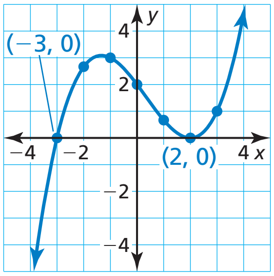
$$f(x) = \frac{1}{6}(x + 3)(x - 2)^2.$$

SOLUTION

Step 1 Plot the *x*-intercepts. Because -3 and 2 are zeros of f, plot $(-3, 0)$ and $(2, 0)$.

Step 2 Plot points between and beyond the *x*-intercepts.

x	-2	-1	0	1	3
y	$\frac{8}{3}$	3	2	$\frac{2}{3}$	1



Step 3 Determine end behavior. Because $f(x)$ has three factors of the form $x - k$ and a constant factor of $\frac{1}{6}$, f is a cubic function with a positive leading coefficient. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Step 4 Draw the graph so that it passes through the plotted points and has the appropriate end behavior.

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Graph the function.

1. $f(x) = \frac{1}{2}(x + 1)(x - 4)^2$

2. $f(x) = \frac{1}{4}(x + 2)(x - 1)(x - 3)$

The Location Principle

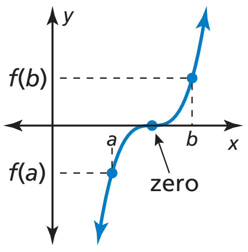
You can use the *Location Principle* to help you find real zeros of polynomial functions.

Core Concept

The Location Principle

If f is a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$, then f has at least one real zero between a and b.

To use this principle to locate real zeros of a polynomial function, find a value a at which the polynomial function is negative and another value b at which the function is positive. You can conclude that the function has *at least* one real zero between a and b.



EXAMPLE 2 Locating Real Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = 6x^3 + 5x^2 - 17x - 6.$$

SOLUTION

Step 1 Use a graphing calculator to make a table.

X	Y1
0	−6
1	−12
2	28
3	150
4	390
5	784
6	1368

X=1

Step 2 Use the Location Principle. From the table shown, you can see that $f(1) < 0$ and $f(2) > 0$. So, by the Location Principle, f has a zero between 1 and 2. Because f is a polynomial function of degree 3, it has three zeros. The only possible *rational* zero between 1 and 2 is $\frac{3}{2}$. Using synthetic division, you can confirm that $\frac{3}{2}$ is a zero.

Step 3 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factor $x - \frac{3}{2}$ gives a quotient of $6x^2 + 14x + 4$. So, you can factor $f(x)$ as

$$f(x) = \left(x - \tfrac{3}{2}\right)(6x^2 + 14x + 4)$$

$$= 2\left(x - \tfrac{3}{2}\right)(3x^2 + 7x + 2)$$

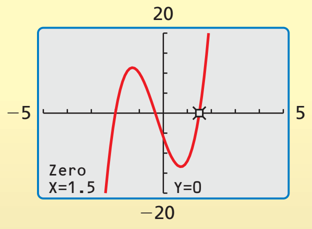
$$= 2\left(x - \tfrac{3}{2}\right)(3x + 1)(x + 2).$$

From the factorization, there are three zeros. The zeros of f are

$\frac{3}{2}$, $-\frac{1}{3}$, and -2.

Check this by graphing f.

Check



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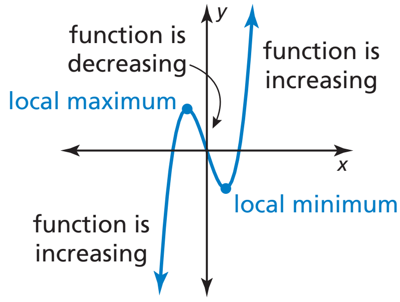
3. Find all real zeros of $f(x) = 18x^3 + 21x^2 - 13x - 6$.

Turning Points

Another important characteristic of graphs of polynomial functions is that they have *turning points* corresponding to local maximum and minimum values.

> **READING**
> Local maximum and local minimum are sometimes referred to as *relative maximum* and *relative minimum*.

- The y-coordinate of a turning point is a **local maximum** of the function when the point is higher than all nearby points.
- The y-coordinate of a turning point is a **local minimum** of the function when the point is lower than all nearby points.



The turning points of a graph help determine the intervals for which a function is increasing or decreasing.

Core Concept

Turning Points of Polynomial Functions

1. The graph of every polynomial function of degree n has *at most* $n - 1$ turning points.
2. If a polynomial function has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

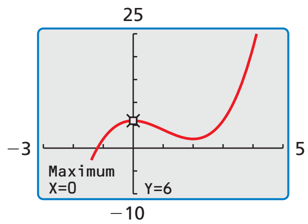
EXAMPLE 3 Finding Turning Points

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

a. $f(x) = x^3 - 3x^2 + 6$ **b.** $g(x) = x^4 - 6x^3 + 3x^2 + 10x - 3$

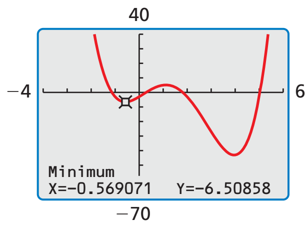
SOLUTION

a. Use a graphing calculator to graph the function. The graph of f has one x-intercept and two turning points. Use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.



▶ The x-intercept of the graph is $x \approx -1.20$. The function has a local maximum at $(0, 6)$ and a local minimum at $(2, 2)$. The function is increasing when $x < 0$ and $x > 2$ and decreasing when $0 < x < 2$.

b. Use a graphing calculator to graph the function. The graph of g has four x-intercepts and three turning points. Use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.



▶ The x-intercepts of the graph are $x \approx -1.14$, $x \approx 0.29$, $x \approx 1.82$, and $x \approx 5.03$. The function has a local maximum at $(1.11, 5.11)$ and local minimums at $(-0.57, -6.51)$ and $(3.96, -43.04)$. The function is increasing when $-0.57 < x < 1.11$ and $x > 3.96$ and decreasing when $x < -0.57$ and $1.11 < x < 3.96$.

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4. Graph $f(x) = 0.5x^3 + x^2 - x + 2$. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

Even and Odd Functions

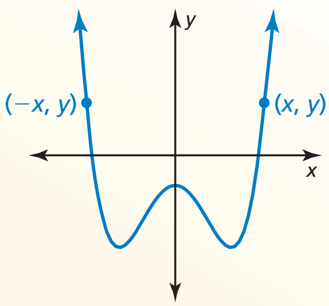
Core Concept

Even and Odd Functions

A function f is an **even function** when $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis*.

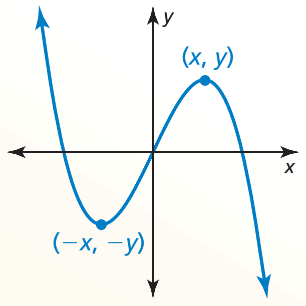
A function f is an **odd function** when $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin*. One way to recognize a graph that is symmetric about the origin is that it looks the same after a 180° rotation about the origin.

Even Function



For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

Odd Function



For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

EXAMPLE 4 Identifying Even and Odd Functions

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = x^3 - 7x$ **b.** $g(x) = x^4 + x^2 - 1$ **c.** $h(x) = x^3 + 2$

SOLUTION

a. Replace x with $-x$ in the equation for f, and then simplify.

$$f(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -f(x)$$

▶ Because $f(-x) = -f(x)$, the function is odd.

b. Replace x with $-x$ in the equation for g, and then simplify.

$$g(-x) = (-x)^4 + (-x)^2 - 1 = x^4 + x^2 - 1 = g(x)$$

▶ Because $g(-x) = g(x)$, the function is even.

c. Replacing x with $-x$ in the equation for h produces

$$h(-x) = (-x)^3 + 2 = -x^3 + 2.$$

▶ Because $h(x) = x^3 + 2$ and $-h(x) = -x^3 - 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

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Determine whether the function is *even*, *odd*, or *neither*.

5. $f(x) = -x^2 + 5$ **6.** $f(x) = x^4 - 5x^3$ **7.** $f(x) = 2x^5$

4.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A local maximum or local minimum of a polynomial function occurs at a ______________ point of the graph of the function.

2. **WRITING** Explain what a local maximum of a function is and how it may be different from the maximum value of the function.

Monitoring Progress and Modeling with Mathematics

ANALYZING RELATIONSHIPS In Exercises 3–6, match the function with its graph.

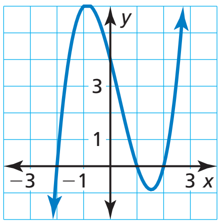
3. $f(x) = (x - 1)(x - 2)(x + 2)$

4. $h(x) = (x + 2)^2(x + 1)$

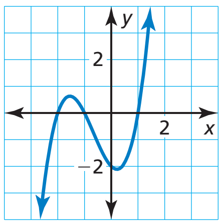
5. $g(x) = (x + 1)(x - 1)(x + 2)$

6. $f(x) = (x - 1)^2(x + 2)$

A.



B.



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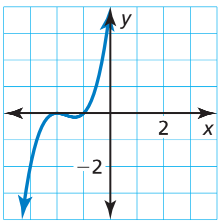

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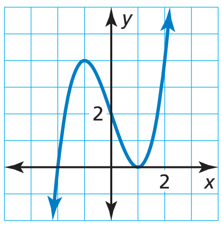


In Exercises 7–14, graph the function. *(See Example 1.)*

7. $f(x) = (x - 2)^2(x + 1)$

8. $f(x) = (x + 2)^2(x + 4)^2$

9. $h(x) = (x + 1)^2(x - 1)(x - 3)$

10. $g(x) = 4(x + 1)(x + 2)(x - 1)$

11. $h(x) = \frac{1}{3}(x - 5)(x + 2)(x - 3)$

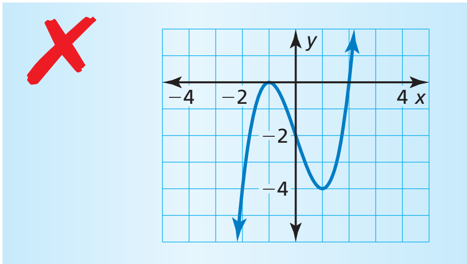
12. $g(x) = \frac{1}{12}(x + 4)(x + 8)(x - 1)$

13. $h(x) = (x - 3)(x^2 + x + 1)$

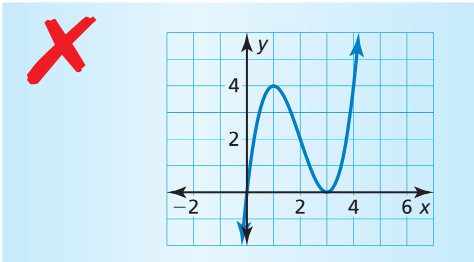
14. $f(x) = (x - 4)(2x^2 - 2x + 1)$

ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in using factors to graph f.

15. $f(x) = (x + 2)(x - 1)^2$



16. $f(x) = x^2(x - 3)^3$



In Exercises 17–22, find all real zeros of the function. *(See Example 2.)*

17. $f(x) = x^3 - 4x^2 - x + 4$

18. $f(x) = x^3 - 3x^2 - 4x + 12$

19. $h(x) = 2x^3 + 7x^2 - 5x - 4$

20. $h(x) = 4x^3 - 2x^2 - 24x - 18$

21. $g(x) = 4x^3 + x^2 - 51x + 36$

22. $f(x) = 2x^3 - 3x^2 - 32x - 15$

In Exercises 23–30, graph the function. Identify the *x*-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. *(See Example 3.)*

23. $g(x) = 2x^3 + 8x^2 - 3$

24. $g(x) = -x^4 + 3x$

25. $h(x) = x^4 - 3x^2 + x$

26. $f(x) = x^5 - 4x^3 + x^2 + 2$

27. $f(x) = 0.5x^3 - 2x + 2.5$

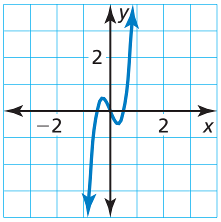
28. $f(x) = 0.7x^4 - 3x^3 + 5x$

29. $h(x) = x^5 + 2x^2 - 17x - 4$

30. $g(x) = x^4 - 5x^3 + 2x^2 + x - 3$

In Exercises 31–36, estimate the coordinates of each turning point. State whether each corresponds to a local maximum or a local minimum. Then estimate the real zeros and find the least possible degree of the function.

31.



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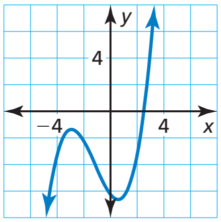

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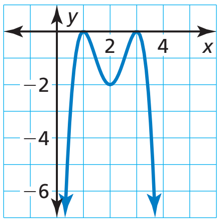

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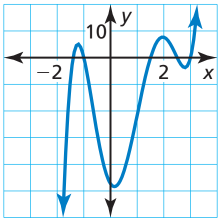

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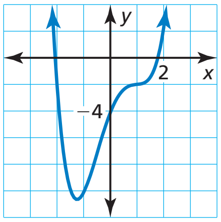

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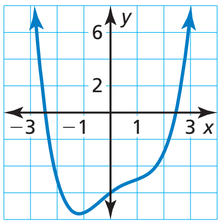


OPEN-ENDED **In Exercises 37 and 38, sketch a graph of a polynomial function *f* having the given characteristics.**

37.
- The graph of f has x-intercepts at $x = -4$, $x = 0$, and $x = 2$.
- f has a local maximum value when $x = 1$.
- f has a local minimum value when $x = -2$.

38.
- The graph of f has x-intercepts at $x = -3$, $x = 1$, and $x = 5$.
- f has a local maximum value when $x = 1$.
- f has a local minimum value when $x = -2$ and when $x = 4$.

In Exercises 39–46, determine whether the function is *even, odd,* or *neither*. *(See Example 4.)*

39. $h(x) = 4x^7$

40. $g(x) = -2x^6 + x^2$

41. $f(x) = x^4 + 3x^2 - 2$

42. $f(x) = x^5 + 3x^3 - x$

43. $g(x) = x^2 + 5x + 1$

44. $f(x) = -x^3 + 2x - 9$

45. $f(x) = x^4 - 12x^2$

46. $h(x) = x^5 + 3x^4$

47. **USING TOOLS** When a swimmer does the breaststroke, the function

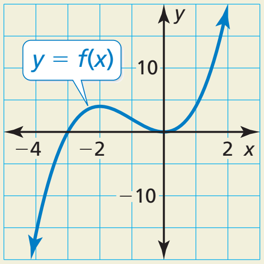
$$S = -241t^7 + 1060t^6 - 1870t^5 + 1650t^4 - 737t^3 + 144t^2 - 2.43t$$

models the speed S (in meters per second) of the swimmer during one complete stroke, where t is the number of seconds since the start of the stroke and $0 \le t \le 1.22$. Use a graphing calculator to graph the function. At what time during the stroke is the swimmer traveling the fastest?

48. **USING TOOLS** During a recent period of time, the number S (in thousands) of students enrolled in public schools in a certain country can be modeled by $S = 1.64x^3 - 102x^2 + 1710x + 36{,}300$, where x is time (in years). Use a graphing calculator to graph the function for the interval $0 \le x \le 41$. Then describe how the public school enrollment changes over this period of time.

49. **WRITING** Why is the adjective *local*, used to describe the maximums and minimums of cubic functions, sometimes not required for quadratic functions?

50. **HOW DO YOU SEE IT?** The graph of a polynomial function is shown.



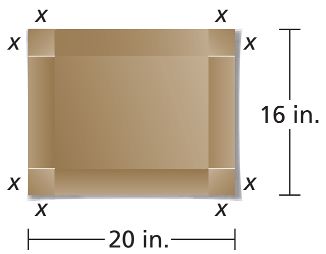
a. Find the zeros, local maximum, and local minimum values of the function.

b. Compare the x-intercepts of the graphs of $y = f(x)$ and $y = -f(x)$.

c. Compare the maximum and minimum values of the functions $y = f(x)$ and $y = -f(x)$.

51. **MAKING AN ARGUMENT** Your friend claims that the product of two odd functions is an odd function. Is your friend correct? Explain your reasoning.

52. **MODELING WITH MATHEMATICS** You are making a rectangular box out of a 16-inch-by-20-inch piece of cardboard. The box will be formed by making the cuts shown in the diagram and folding up the sides. You want the box to have the greatest volume possible.



a. How long should you make the cuts?

b. What is the maximum volume?

c. What are the dimensions of the finished box?

53. **PROBLEM SOLVING** Quonset huts are temporary, all-purpose structures shaped like half-cylinders. You have 1100 square feet of material to build a quonset hut.

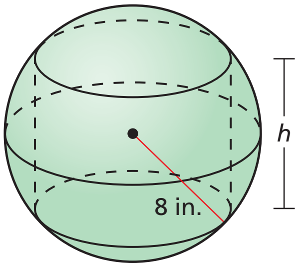
a. The surface area S of a quonset hut is given by $S = \pi r^2 + \pi r\ell$. Substitute 1100 for S and then write an expression for ℓ in terms of r.

b. The volume V of a quonset hut is given by $V = \frac{1}{2}\pi r^2\ell$. Write an equation that gives V as a function in terms of r only.

c. Find the value of r that maximizes the volume of the hut.

54. **THOUGHT PROVOKING** Write and graph a polynomial function that has one real zero in each of the intervals $-2 < x < -1$, $0 < x < 1$, and $4 < x < 5$. Is there a maximum degree that such a polynomial function can have? Justify your answer.

55. **MATHEMATICAL CONNECTIONS** A cylinder is inscribed in a sphere of radius 8 inches. Write an equation for the volume of the cylinder as a function of h. Find the value of h that maximizes the volume of the inscribed cylinder. What is the maximum volume of the cylinder?



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

State whether the table displays *linear data*, *quadratic data*, or *neither*. Explain. *(Section 2.4)*

56.

Months, x	0	1	2	3
Savings (dollars), y	100	150	200	250

57.

Time (seconds), x	0	1	2	3
Height (feet), y	300	284	236	156