4.9 Modeling with Polynomial Functions

Essential Question How can you find a polynomial model for real-life data?

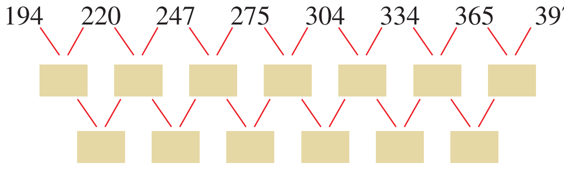
EXPLORATION 1 **Modeling Real-Life Data**

Work with a partner. The distance a baseball travels after it is hit depends on the angle at which it was hit and the initial speed. The table shows the distances a baseball hit at an angle of 35° travels at various initial speeds.

Initial speed, *x* (miles per hour)	80	85	90	95	100	105	110	115
Distance, *y* (feet)	194	220	247	275	304	334	365	397

a. Recall that when data have equally-spaced *x*-values, you can analyze patterns in the differences of the *y*-values to determine what type of function can be used to model the data. If the first differences are constant, then the set of data fits a linear model. If the second differences are constant, then the set of data fits a quadratic model.

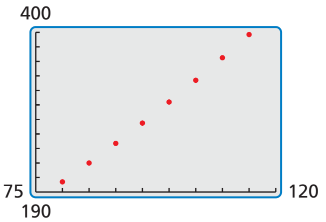
Find the first and second differences of the data. Are the data linear or quadratic? Explain your reasoning.



USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

b. Use a graphing calculator to draw a scatter plot of the data. Do the data appear linear or quadratic? Use the *regression* feature of the graphing calculator to find a linear or quadratic model that best fits the data.



c. Use the model you found in part (b) to find the distance a baseball travels when it is hit at an angle of 35° and travels at an initial speed of 120 miles per hour.

d. According to the *Baseball Almanac*, "Any drive over 400 feet is noteworthy. A blow of 450 feet shows exceptional power, as the majority of major league players are unable to hit a ball that far. Anything in the 500-foot range is genuinely historic." Estimate the initial speed of a baseball that travels a distance of 500 feet.

Communicate Your Answer

2. How can you find a polynomial model for real-life data?

3. How well does the model you found in Exploration 1(b) fit the data? Do you think the model is valid for any initial speed? Explain your reasoning.

4.9 Lesson

Core Vocabulary

finite differences, *p. 220*

Previous

scatter plot

What You Will Learn

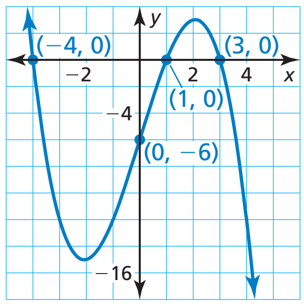
- Write polynomial functions for sets of points.
- Write polynomial functions using finite differences.
- Use technology to find models for data sets.

Writing Polynomial Functions for a Set of Points

You know that two points determine a line and three points not on a line determine a parabola. In Example 1, you will see that four points not on a line or a parabola determine the graph of a cubic function.

EXAMPLE 1 Writing a Cubic Function

Write the cubic function whose graph is shown.



SOLUTION

Step 1 Use the three x-intercepts to write the function in factored form.

$$f(x) = a(x + 4)(x - 1)(x - 3)$$

Step 2 Find the value of a by substituting the coordinates of the point $(0, -6)$.

$$-6 = a(0 + 4)(0 - 1)(0 - 3)$$

$$-6 = 12a$$

$$-\frac{1}{2} = a$$

▶ The function is $f(x) = -\frac{1}{2}(x + 4)(x - 1)(x - 3)$.

Check

Check the end behavior of f. The degree of f is odd and $a < 0$. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$, which matches the graph. ✓

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Write a cubic function whose graph passes through the given points.

1. $(-4, 0), (0, 10), (2, 0), (5, 0)$

2. $(-1, 0), (0, -12), (2, 0), (3, 0)$

Finite Differences

When the x-values in a data set are equally spaced, the differences of consecutive y-values are called **finite differences**. Recall from Section 2.4 that the first and second differences of $y = x^2$ are:

equally-spaced x-values

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

first differences: −5 −3 −1 1 3 5

second differences: 2 2 2 2 2

Notice that $y = x^2$ has degree *two* and that the *second* differences are constant and nonzero. This illustrates the first of the two properties of finite differences shown on the next page.

Core Concept

Properties of Finite Differences

1. If a polynomial function $y = f(x)$ has degree n, then the nth differences of function values for equally-spaced x-values are nonzero and constant.
2. Conversely, if the nth differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree n.

The second property of finite differences allows you to write a polynomial function that models a set of equally-spaced data.

EXAMPLE 2 Writing a Function Using Finite Differences

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	1	2	3	4	5	6	7
$f(x)$	1	4	10	20	35	56	84

SOLUTION

Step 1 Write the function values. Find the first differences by subtracting consecutive values. Then find the second differences by subtracting consecutive first differences. Continue until you obtain differences that are nonzero and constant.

$f(1)$ $f(2)$ $f(3)$ $f(4)$ $f(5)$ $f(6)$ $f(7)$

1 4 10 20 35 56 84 — Write function values for equally-spaced x-values.

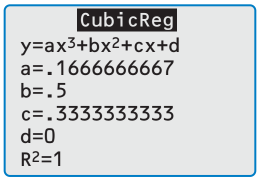
3 6 10 15 21 28 — First differences

3 4 5 6 7 — Second differences

1 1 1 1 — Third differences

Because the third differences are nonzero and constant, you can model the data *exactly* with a cubic function.

Step 2 Enter the data into a graphing calculator and use cubic regression to obtain a polynomial function.



▶ Because $\frac{1}{6} \approx 0.1666666667$, $\frac{1}{2} = 0.5$, and $\frac{1}{3} \approx 0.333333333$, a polynomial function that fits the data exactly is

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x.$$

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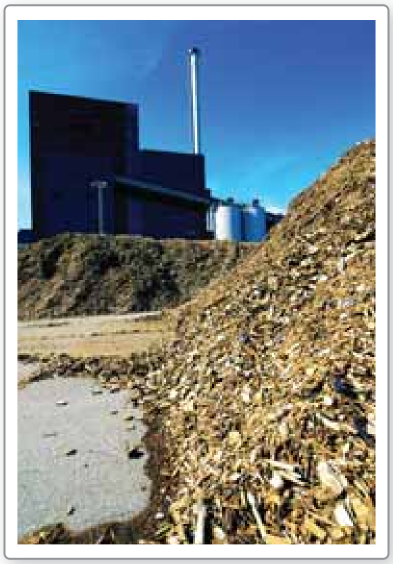
3. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	-3	-2	-1	0	1	2
$f(x)$	6	15	22	21	6	-29

Finding Models Using Technology

In Examples 1 and 2, you found a cubic model that *exactly* fits a set of data. In many real-life situations, you cannot find models to fit data exactly. Despite this limitation, you can still use technology to approximate the data with a polynomial model, as shown in the next example.

EXAMPLE 3 Real-Life Application



According to the U.S. Department of Energy, *biomass* includes "agricultural and forestry residues, municipal solid wastes, industrial wastes, and terrestrial and aquatic crops grown solely for energy purposes." Among the uses for biomass is production of electricity and liquid fuels such as ethanol.

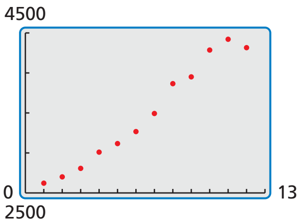
The table shows the total U.S. biomass energy consumptions y (in trillions of British thermal units, or Btus) in the year t, where $t = 1$ corresponds to 2001. Find a polynomial model for the data. Use the model to estimate the total U.S. biomass energy consumption in 2013.

t	1	2	3	4	5	6
y	2622	2701	2807	3010	3117	3267

t	7	8	9	10	11	12
y	3493	3866	3951	4286	4421	4316

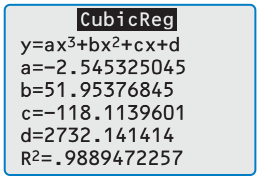
SOLUTION

Step 1 Enter the data into a graphing calculator and make a scatter plot. The data suggest a cubic model.

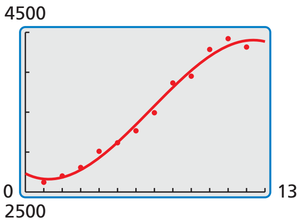


Step 2 Use the *cubic regression* feature. The polynomial model is

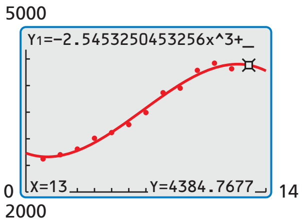
$$y = -2.545t^3 + 51.95t^2 - 118.1t + 2732.$$



Step 3 Check the model by graphing it and the data in the same viewing window.



Step 4 Use the *trace* feature to estimate the value of the model when $t = 13$.



▶ The approximate total U.S. biomass energy consumption in 2013 was about 4385 trillion Btus.

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Use a graphing calculator to find a polynomial function that fits the data.

4.

x	1	2	3	4	5	6
y	5	13	17	11	11	56

5.

x	0	2	4	6	8	10
y	8	0	15	69	98	87

4.9 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

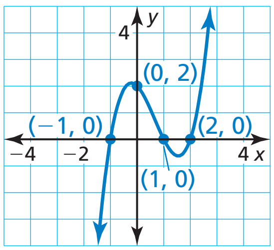
1. **COMPLETE THE SENTENCE** When the x-values in a set of data are equally spaced, the differences of consecutive y-values are called ________________.

2. **WRITING** Explain how you know when a set of data could be modeled by a cubic function.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, write a cubic function whose graph is shown. *(See Example 1.)*

3.



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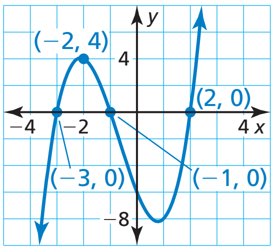

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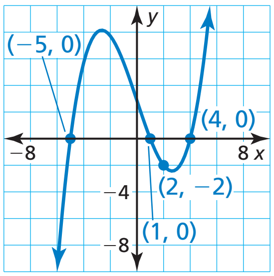

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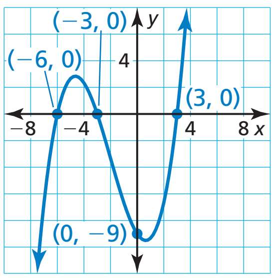


In Exercises 7–12, use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. *(See Example 2.)*

7.

x	−6	−3	0	3	6	9
$f(x)$	−2	15	−4	49	282	803

8.

x	−1	0	1	2	3	4
$f(x)$	−14	−5	−2	7	34	91

9. (−4, −317), (−3, −37), (−2, 21), (−1, 7), (0, −1), (1, 3), (2, −47), (3, −289), (4, −933)

10. (−6, 744), (−4, 154), (−2, 4), (0, −6), (2, 16), (4, 154), (6, 684), (8, 2074), (10, 4984)

11. (−2, 968), (−1, 422), (0, 142), (1, 26), (2, −4), (3, −2), (4, 2), (5, 2), (6, 16)

12. (1, 0), (2, 6), (3, 2), (4, 6), (5, 12), (6, −10), (7, −114), (8, −378), (9, −904)

13. **ERROR ANALYSIS** Describe and correct the error in writing a cubic function whose graph passes through the given points.

(−6, 0), (1, 0), (3, 0), (0, 54)

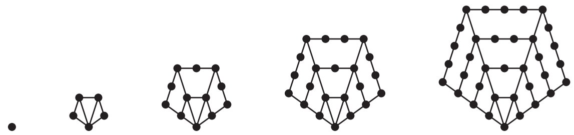
$54 = a(0 - 6)(0 + 1)(0 + 3)$

$54 = -18a$

$a = -3$

$f(x) = -3(x - 6)(x + 1)(x + 3)$

14. **MODELING WITH MATHEMATICS** The dot patterns show pentagonal numbers. The number of dots in the nth pentagonal number is given by $f(n) = \frac{1}{2}n(3n - 1)$. Show that this function has constant second-order differences.



15. **OPEN-ENDED** Write three different cubic functions that pass through the points (3, 0), (4, 0), and (2, 6). Justify your answers.

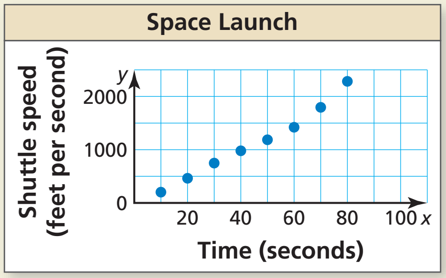
16. **MODELING WITH MATHEMATICS** The table shows the ages of cats and their corresponding ages in human years. Find a polynomial model for the data for the first 8 years of a cat's life. Use the model to estimate the age (in human years) of a cat that is 3 years old. *(See Example 3.)*

Age of cat, x	1	2	4	6	7	8
Human years, y	15	24	32	40	44	48

17. MODELING WITH MATHEMATICS The data in the table show the average speeds y (in miles per hour) of a pontoon boat for several different engine speeds x (in hundreds of revolutions per minute, or RPMs). Find a polynomial model for the data. Estimate the average speed of the pontoon boat when the engine speed is 2800 RPMs.

x	10	20	25	30	45	55
y	4.5	8.9	13.8	18.9	29.9	37.7

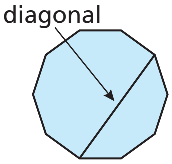
18. HOW DO YOU SEE IT? The graph shows typical speeds y (in feet per second) of a space shuttle x seconds after it is launched.



a. What type of polynomial function models the data? Explain.

b. Which nth-order finite difference should be constant for the function in part (a)? Explain.

19. MATHEMATICAL CONNECTIONS The table shows the number of diagonals for polygons with n sides. Find a polynomial function that fits the data. Determine the total number of diagonals in the decagon shown.



Number of sides, n	3	4	5	6	7	8
Number of diagonals, d	0	2	5	9	14	20

20. MAKING AN ARGUMENT Your friend states that it is not possible to determine the degree of a function given the first-order differences. Is your friend correct? Explain your reasoning.

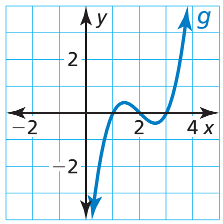
21. WRITING Explain why you cannot always use finite differences to find a model for real-life data sets.

22. THOUGHT PROVOKING A, B, and C are zeros of a cubic polynomial function. Choose values for A, B, and C such that the distance from A to B is less than or equal to the distance from A to C. Then write the function using the A, B, and C values you chose.

23. MULTIPLE REPRESENTATIONS Order the polynomial functions according to their degree, from least to greatest.

A. $f(x) = -3x + 2x^2 + 1$

B.



C.

x	−2	−1	0	1	2	3
$h(x)$	8	6	4	2	0	−2

D.

x	−2	−1	0	1	2	3
$k(x)$	25	6	7	4	−3	10

24. ABSTRACT REASONING Substitute the expressions $z, z + 1, z + 2, \ldots, z + 5$ for x in the function $f(x) = ax^3 + bx^2 + cx + d$ to generate six equally-spaced ordered pairs. Then show that the third-order differences are constant.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation using square roots. *(Section 3.1)*

25. $x^2 - 6 = 30$

26. $5x^2 - 38 = 187$

27. $2(x - 3)^2 = 24$

28. $\frac{4}{3}(x + 5)^2 = 4$

Solve the equation using the Quadratic Formula. *(Section 3.4)*

29. $2x^2 + 3x = 5$

30. $2x^2 + \frac{1}{2} = 2x$

31. $2x^2 + 3x = -3x^2 + 1$

32. $4x - 20 = x^2$