5.1 *n*th Roots and Rational Exponents

Essential Question How can you use a rational exponent to represent a power involving a radical?

Previously, you learned that the nth root of a can be represented as

$\sqrt[n]{a} = a^{1/n}$ Definition of rational exponent

for any real number a and integer n greater than 1.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated definitions and previously established results.

EXPLORATION 1 Exploring the Definition of a Rational Exponent

Work with a partner. Use a calculator to show that each statement is true.

a. $\sqrt{9} = 9^{1/2}$ **b.** $\sqrt{2} = 2^{1/2}$ **c.** $\sqrt[3]{8} = 8^{1/3}$

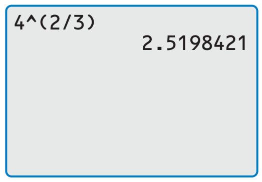
d. $\sqrt[3]{3} = 3^{1/3}$ **e.** $\sqrt[4]{16} = 16^{1/4}$ **f.** $\sqrt[4]{12} = 12^{1/4}$

EXPLORATION 2 Writing Expressions in Rational Exponent Form

Work with a partner. Use the definition of a rational exponent and the properties of exponents to write each expression as a base with a single rational exponent. Then use a calculator to evaluate each expression. Round your answer to two decimal places.

Sample

$$\left(\sqrt[3]{4}\right)^2 = (4^{1/3})^2$$
$$= 4^{2/3}$$
$$\approx 2.52$$



a. $\left(\sqrt{5}\right)^3$ **b.** $\left(\sqrt[4]{4}\right)^2$ **c.** $\left(\sqrt[3]{9}\right)^2$

d. $\left(\sqrt[5]{10}\right)^4$ **e.** $\left(\sqrt{15}\right)^3$ **f.** $\left(\sqrt[3]{27}\right)^4$

EXPLORATION 3 Writing Expressions in Radical Form

Work with a partner. Use the properties of exponents and the definition of a rational exponent to write each expression as a radical raised to an exponent. Then use a calculator to evaluate each expression. Round your answer to two decimal places.

Sample $5^{2/3} = (5^{1/3})^2 = \left(\sqrt[3]{5}\right)^2 \approx 2.92$

a. $8^{2/3}$ **b.** $6^{5/2}$ **c.** $12^{3/4}$

d. $10^{3/2}$ **e.** $16^{3/2}$ **f.** $20^{6/5}$

Communicate Your Answer

4. How can you use a rational exponent to represent a power involving a radical?

5. Evaluate each expression *without* using a calculator. Explain your reasoning.

a. $4^{3/2}$ **b.** $32^{4/5}$ **c.** $625^{3/4}$

d. $49^{3/2}$ **e.** $125^{4/3}$ **f.** $100^{6/3}$

5.1 Lesson

Core Vocabulary

*n*th root of *a*, *p. 238*
index of a radical, *p. 238*

Previous
square root
cube root
exponent

What You Will Learn

- Find *n*th roots of numbers.
- Evaluate expressions with rational exponents.
- Solve equations using *n*th roots.

*n*th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$. In general, for an integer n greater than 1, if $b^n = a$, then b is an **nth root of a**. An nth root of a is written as $\sqrt[n]{a}$, where n is the **index** of the radical.

You can also write an nth root of a as a power of a. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$

$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$

$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

UNDERSTANDING MATHEMATICAL TERMS

When n is even and $a > 0$, there are two real roots. The positive root is called the *principal root*.

Core Concept

Real *n*th Roots of *a*

Let n be an integer $(n > 1)$ and let a be a real number.

	n is an even integer.		**n is an odd integer.**
$a < 0$	No real nth roots	$a < 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$
$a = 0$	One real nth root: $\sqrt[n]{0} = 0$	$a = 0$	One real nth root: $\sqrt[n]{0} = 0$
$a > 0$	Two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$	$a > 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$

EXAMPLE 1 Finding *n*th Roots

Find the indicated real nth root(s) of a.

a. $n = 3$, $a = -216$

b. $n = 4$, $a = 81$

SOLUTION

a. Because $n = 3$ is odd and $a = -216 < 0$, -216 has one real cube root. Because $(-6)^3 = -216$, you can write $\sqrt[3]{-216} = -6$ or $(-216)^{1/3} = -6$.

b. Because $n = 4$ is even and $a = 81 > 0$, 81 has two real fourth roots. Because $3^4 = 81$ and $(-3)^4 = 81$, you can write $\pm\sqrt[4]{81} = \pm 3$ or $\pm 81^{1/4} = \pm 3$.

Monitoring Progress

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Find the indicated real nth root(s) of a.

1. $n = 4$, $a = 16$

2. $n = 2$, $a = -49$

3. $n = 3$, $a = -125$

4. $n = 5$, $a = 243$

Rational Exponents

A rational exponent does not have to be of the form $\frac{1}{n}$. Other rational numbers, such as $\frac{3}{2}$ and $-\frac{1}{2}$, can also be used as exponents. Two properties of rational exponents are shown below.

Core Concept

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m}, \ a \neq 0$$

EXAMPLE 2 **Evaluating Expressions with Rational Exponents**

Evaluate each expression.

a. $16^{3/2}$ **b.** $32^{-3/5}$

SOLUTION

Rational Exponent Form	Radical Form
a. $16^{3/2} = (16^{1/2})^3 = 4^3 = 64$	$16^{3/2} = (\sqrt{16})^3 = 4^3 = 64$
b. $32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(32^{1/5})^3} = \frac{1}{2^3} = \frac{1}{8}$	$32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8}$

When using a calculator to approximate an nth root, you may want to rewrite the nth root in rational exponent form.

COMMON ERROR

Be sure to use parentheses to enclose a rational exponent: 9^(1/5) ≈ 1.55. Without them, the calculator evaluates a power and then divides: 9^1/5 = 1.8.

EXAMPLE 3 **Approximating Expressions with Rational Exponents**

Evaluate each expression using a calculator. Round your answer to two decimal places.

a. $9^{1/5}$ **b.** $12^{3/8}$ **c.** $(\sqrt[4]{7})^3$

SOLUTION

a. $9^{1/5} \approx 1.55$

b. $12^{3/8} \approx 2.54$

c. Before evaluating $(\sqrt[4]{7})^3$, rewrite the expression in rational exponent form.

$$(\sqrt[4]{7})^3 = 7^{3/4} \approx 4.30$$

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9^(1/5)
          1.551845574
12^(3/8)
          2.539176951
7^(3/4)
          4.303517071
```

Monitoring Progress

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Evaluate the expression without using a calculator.

5. $4^{5/2}$ **6.** $9^{-1/2}$ **7.** $81^{3/4}$ **8.** $1^{7/8}$

Evaluate the expression using a calculator. Round your answer to two decimal places when appropriate.

9. $6^{2/5}$ **10.** $64^{-2/3}$ **11.** $(\sqrt[4]{16})^5$ **12.** $(\sqrt[3]{-30})^2$

Solving Equations Using nth Roots

To solve an equation of the form $u^n = d$, where u is an algebraic expression, take the nth root of each side.

EXAMPLE 4 Solving Equations Using nth Roots

Find the real solution(s) of (a) $4x^5 = 128$ and (b) $(x - 3)^4 = 21$.

SOLUTION

a. $4x^5 = 128$ — Write original equation.

$x^5 = 32$ — Divide each side by 4.

$x = \sqrt[5]{32}$ — Take fifth root of each side.

$x = 2$ — Simplify.

▶ The solution is $x = 2$.

COMMON ERROR

When n is even and $a > 0$, be sure to consider both the positive and negative nth roots of a.

b. $(x - 3)^4 = 21$ — Write original equation.

$x - 3 = \pm\sqrt[4]{21}$ — Take fourth root of each side.

$x = 3 \pm \sqrt[4]{21}$ — Add 3 to each side.

$x = 3 + \sqrt[4]{21}$ or $x = 3 - \sqrt[4]{21}$ — Write solutions separately.

$x \approx 5.14$ or $x \approx 0.86$ — Use a calculator.

▶ The solutions are $x \approx 5.14$ and $x \approx 0.86$.

EXAMPLE 5 Real-Life Application

A hospital purchases an ultrasound machine for \$50,000. The hospital expects the useful life of the machine to be 10 years, at which time its value will have depreciated to \$8000. The hospital uses the declining balances method for depreciation, so the annual depreciation rate r (in decimal form) is given by the formula

$$r = 1 - \left(\frac{S}{C}\right)^{1/n}.$$

In the formula, n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars). What annual depreciation rate did the hospital use?

SOLUTION

The useful life is 10 years, so $n = 10$. The machine depreciates to \$8000, so $S = 8000$. The original cost is \$50,000, so $C = 50{,}000$. So, the annual depreciation rate is

$$r = 1 - \left(\frac{S}{C}\right)^{1/n} = 1 - \left(\frac{8000}{50{,}000}\right)^{1/10} = 1 - \left(\frac{4}{25}\right)^{1/10} \approx 0.167.$$

▶ The annual depreciation rate is about 0.167, or 16.7%.

Monitoring Progress

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Find the real solution(s) of the equation. Round your answer to two decimal places when appropriate.

13. $8x^3 = 64$ **14.** $\frac{1}{2}x^5 = 512$ **15.** $(x + 5)^4 = 16$ **16.** $(x - 2)^3 = -14$

17. WHAT IF? In Example 5, what is the annual depreciation rate when the salvage value is \$6000?

5.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Rewrite the expression $a^{-s/t}$ in radical form. Then state the index of the radical.

2. **COMPLETE THE SENTENCE** For an integer n greater than 1, if $b^n = a$, then b is a(n) ___________ of a.

3. **WRITING** Explain how to use the sign of a to determine the number of real fourth roots of a and the number of real fifth roots of a.

4. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$(a^{1/n})^m$	$(\sqrt[n]{a})^m$	$(\sqrt[m]{a})^{-n}$	$a^{m/n}$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, find the indicated real *n*th root(s) of *a*. *(See Example 1.)*

5. $n = 3, a = 8$
6. $n = 5, a = -1$
7. $n = 2, a = 0$
8. $n = 4, a = 256$
9. $n = 5, a = -32$
10. $n = 6, a = -729$

In Exercises 11–18, evaluate the expression without using a calculator. *(See Example 2.)*

11. $64^{1/6}$
12. $8^{1/3}$
13. $25^{3/2}$
14. $81^{3/4}$
15. $(-243)^{1/5}$
16. $(-64)^{4/3}$
17. $8^{-2/3}$
18. $16^{-7/4}$

ERROR ANALYSIS **In Exercises 19 and 20, describe and correct the error in evaluating the expression.**

19. ✗
$$\begin{aligned} 27^{2/3} &= (27^{1/3})^2 \\ &= 9^2 \\ &= 81 \end{aligned}$$

20. ✗
$$\begin{aligned} 256^{4/3} &= \left(\sqrt[4]{256}\right)^3 \\ &= 4^3 \\ &= 64 \end{aligned}$$

USING STRUCTURE **In Exercises 21–24, match the equivalent expressions. Explain your reasoning.**

21. $(\sqrt[3]{5})^4$
22. $(\sqrt[4]{5})^3$
23. $\dfrac{1}{\sqrt[4]{5}}$
24. $-\sqrt[4]{5}$

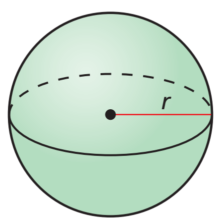
A. $5^{-1/4}$
B. $5^{4/3}$
C. $-5^{1/4}$
D. $5^{3/4}$

In Exercises 25–32, evaluate the expression using a calculator. Round your answer to two decimal places when appropriate. *(See Example 3.)*

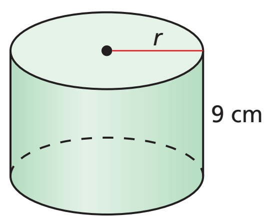
25. $\sqrt[5]{32{,}768}$
26. $\sqrt[7]{1695}$
27. $25^{-1/3}$
28. $85^{1/6}$
29. $20{,}736^{4/5}$
30. $86^{-5/6}$
31. $(\sqrt[4]{187})^3$
32. $(\sqrt[5]{-8})^8$

MATHEMATICAL CONNECTIONS **In Exercises 33 and 34, find the radius of the figure with the given volume.**

33. $V = 216\text{ ft}^3$



34. $V = 1332\text{ cm}^3$



In Exercises 35–44, find the real solution(s) of the equation. Round your answer to two decimal places when appropriate. *(See Example 4.)*

35. $x^3 = 125$

36. $5x^3 = 1080$

37. $(x + 10)^5 = 70$

38. $(x - 5)^4 = 256$

39. $x^5 = -48$

40. $7x^4 = 56$

41. $x^6 + 36 = 100$

42. $x^3 + 40 = 25$

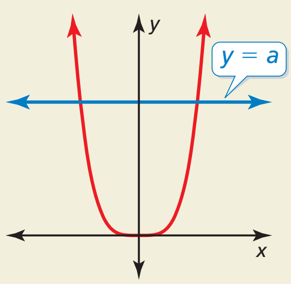
43. $\frac{1}{3}x^4 = 27$

44. $\frac{1}{6}x^3 = -36$

45. MODELING WITH MATHEMATICS When the average price of an item increases from p_1 to p_2 over a period of n years, the annual rate of inflation r (in decimal form) is given by $r = \left(\frac{p_2}{p_1}\right)^{1/n} - 1$. Find the rate of inflation for each item in the table. *(See Example 5.)*

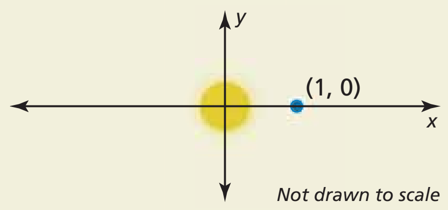
Item	Price in 1913	Price in 2013
Potatoes (lb)	\$0.016	\$0.627
Ham (lb)	\$0.251	\$2.693
Eggs (dozen)	\$0.373	\$1.933

46. HOW DO YOU SEE IT? The graph of $y = x^n$ is shown in red. What can you conclude about the value of n? Determine the number of real nth roots of a. Explain your reasoning.

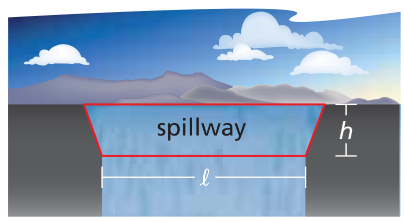


47. NUMBER SENSE Between which two consecutive integers does $\sqrt[4]{125}$ lie? Explain your reasoning.

48. THOUGHT PROVOKING In 1619, Johannes Kepler published his third law, which can be given by $d^3 = t^2$, where d is the mean distance (in astronomical units) of a planet from the Sun and t is the time (in years) it takes the planet to orbit the Sun. It takes Mars 1.88 years to orbit the Sun. Graph a possible location of Mars. Justify your answer. (The diagram shows the Sun at the origin of the xy-plane and a possible location of Earth.)



49. PROBLEM SOLVING A *weir* is a dam that is built across a river to regulate the flow of water. The flow rate Q (in cubic feet per second) can be calculated using the formula $Q = 3.367\ell h^{3/2}$, where ℓ is the length (in feet) of the bottom of the spillway and h is the depth (in feet) of the water on the spillway. Determine the flow rate of a weir with a spillway that is 20 feet long and has a water depth of 5 feet.



50. REPEATED REASONING The mass of the particles that a river can transport is proportional to the sixth power of the speed of the river. A certain river normally flows at a speed of 1 meter per second. What must its speed be in order to transport particles that are twice as massive as usual? 10 times as massive? 100 times as massive?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. Write your answer using only positive exponents. *(Skills Review Handbook)*

51. $5 \cdot 5^4$

52. $\frac{4^2}{4^7}$

53. $(z^2)^{-3}$

54. $\left(\frac{3x}{2}\right)^4$

Write the number in standard form. *(Skills Review Handbook)*

55. 5×10^3

56. 4×10^{-2}

57. 8.2×10^{-1}

58. 6.93×10^6