5.3 Graphing Radical Functions

Essential Question How can you identify the domain and range of a radical function?

EXPLORATION 1 Identifying Graphs of Radical Functions

Work with a partner. Match each function with its graph. Explain your reasoning. Then identify the domain and range of each function.

a. $f(x) = \sqrt{x}$ **b.** $f(x) = \sqrt[3]{x}$ **c.** $f(x) = \sqrt[4]{x}$ **d.** $f(x) = \sqrt[5]{x}$

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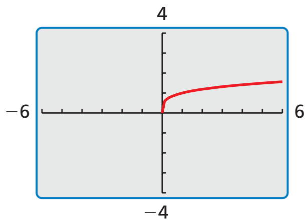

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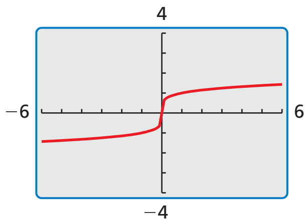

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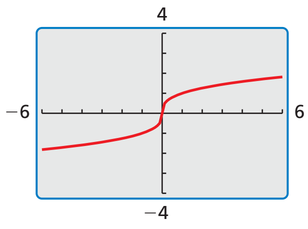

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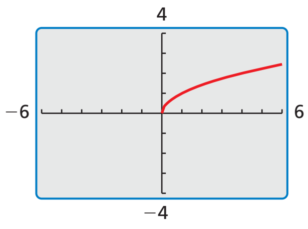
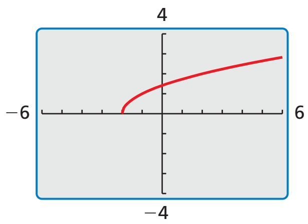


EXPLORATION 2 Identifying Graphs of Transformations

Work with a partner. Match each transformation of $f(x) = \sqrt{x}$ with its graph. Explain your reasoning. Then identify the domain and range of each function.

a. $g(x) = \sqrt{x + 2}$ **b.** $g(x) = \sqrt{x - 2}$ **c.** $g(x) = \sqrt{x + 2} - 2$ **d.** $g(x) = -\sqrt{x + 2}$

A.


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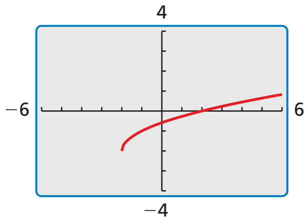

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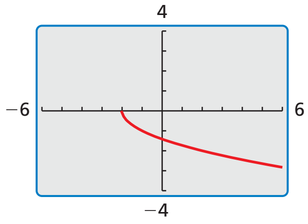

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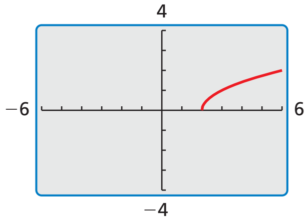


LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

3. How can you identify the domain and range of a radical function?

4. Use the results of Exploration 1 to describe how the domain and range of a radical function are related to the index of the radical.

5.3 Lesson

Core Vocabulary

radical function, *p. 252*

Previous
transformations
parabola
circle

What You Will Learn

- Graph radical functions.
- Write transformations of radical functions.
- Graph parabolas and circles.

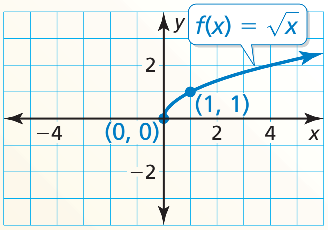
Graphing Radical Functions

A **radical function** contains a radical expression with the independent variable in the radicand. When the radical is a square root, the function is called a *square root function*. When the radical is a cube root, the function is called a *cube root function*.

Core Concept

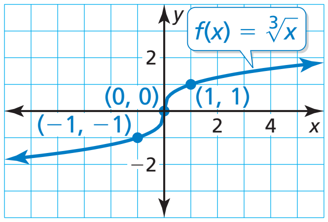
Parent Functions for Square Root and Cube Root Functions

The parent function for the family of square root functions is $f(x) = \sqrt{x}$.



Domain: $x \geq 0$, Range: $y \geq 0$

The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$.



Domain and range: All real numbers

STUDY TIP

A *power function* has the form $y = ax^b$, where a is a real number and b is a rational number. Notice that the parent square root function is a power function, where $a = 1$ and $b = \frac{1}{2}$.

EXAMPLE 1 Graphing Radical Functions

Graph each function. Identify the domain and range of each function.

a. $f(x) = \sqrt{\frac{1}{4}x}$

b. $g(x) = -3\sqrt[3]{x}$

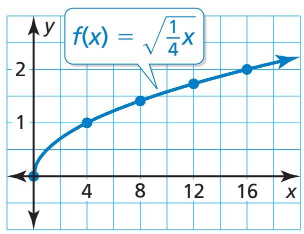
LOOKING FOR STRUCTURE

Example 1(a) uses x-values that are multiples of 4 so that the radicand is an integer.

SOLUTION

a. Make a table of values and sketch the graph.

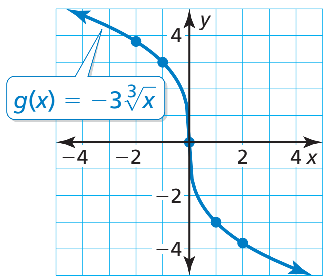
x	0	4	8	12	16
y	0	1	1.41	1.73	2



- The radicand of a square root must be nonnegative. So, the domain is $x \geq 0$. The range is $y \geq 0$.

b. Make a table of values and sketch the graph.

x	−2	−1	0	1	2
y	3.78	3	0	−3	−3.78



- The radicand of a cube root can be any real number. So, the domain and range are all real numbers.

In Example 1, notice that the graph of f is a horizontal stretch of the graph of the parent square root function. The graph of g is a vertical stretch and a reflection in the x-axis of the graph of the parent cube root function. You can transform graphs of radical functions in the same way you transformed graphs of functions previously.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = \sqrt{x-2}$ $g(x) = \sqrt{x+3}$	2 units right 3 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = \sqrt{x} + 7$ $g(x) = \sqrt{x} - 1$	7 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = \sqrt{-x}$ $g(x) = -\sqrt{x}$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = \sqrt{3x}$ $g(x) = \sqrt{\frac{1}{2}x}$	shrink by a factor of $\frac{1}{3}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 4\sqrt{x}$ $g(x) = \frac{1}{5}\sqrt{x}$	stretch by a factor of 4 shrink by a factor of $\frac{1}{5}$

EXAMPLE 2 Transforming Radical Functions

Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x-3} + 4$

b. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{-8x}$

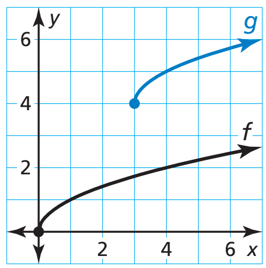
LOOKING FOR STRUCTURE

In Example 2(b), you can use the Product Property of Radicals to write $g(x) = -2\sqrt[3]{x}$. So, you can also describe the graph of g as a vertical stretch by a factor of 2 and a reflection in the x-axis of the graph of f.

SOLUTION

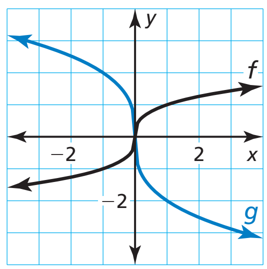
a. Notice that the function is of the form $g(x) = \sqrt{x-h} + k$, where $h = 3$ and $k = 4$.

So, the graph of g is a translation 3 units right and 4 units up of the graph of f.



b. Notice that the function is of the form $g(x) = \sqrt[3]{ax}$, where $a = -8$.

So, the graph of g is a horizontal shrink by a factor of $\frac{1}{8}$ and a reflection in the y-axis of the graph of f.



Monitoring Progress

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1. Graph $g(x) = \sqrt{x+1}$. Identify the domain and range of the function.

2. Describe the transformation of $f(x) = \sqrt[3]{x}$ represented by $g(x) = -\sqrt[3]{x} - 2$. Then graph each function.

Writing Transformations of Radical Functions

EXAMPLE 3 Modeling with Mathematics

Self-Portrait of NASA's Mars Rover Curiosity

The function $E(d) = 0.25\sqrt{d}$ approximates the number of seconds it takes a dropped object to fall d feet on Earth. The function $M(d) = 1.6 \cdot E(d)$ approximates the number of seconds it takes a dropped object to fall d feet on Mars. Write a rule for M. How long does it take a dropped object to fall 64 feet on Mars?

SOLUTION

1. **Understand the Problem** You are given a function that represents the number of seconds it takes a dropped object to fall d feet on Earth. You are asked to write a similar function for Mars and then evaluate the function for a given input.
2. **Make a Plan** Multiply $E(d)$ by 1.6 to write a rule for M. Then find $M(64)$.
3. **Solve the Problem** $M(d) = 1.6 \cdot E(d)$

   $= 1.6 \cdot 0.25\sqrt{d}$ — Substitute $0.25\sqrt{d}$ for $E(d)$.

   $= 0.4\sqrt{d}$ — Simplify.

   Next, find $M(64)$.

   $$M(64) = 0.4\sqrt{64} = 0.4(8) = 3.2$$

   ▶ It takes a dropped object about 3.2 seconds to fall 64 feet on Mars.
4. **Look Back** Use the original functions to check your solution.

   $E(64) = 0.25\sqrt{64} = 2$ $M(64) = 1.6 \cdot E(64) = 1.6 \cdot 2 = 3.2$ ✓

EXAMPLE 4 Writing a Transformed Radical Function

Let the graph of g be a horizontal shrink by a factor of $\frac{1}{6}$ followed by a translation 3 units to the left of the graph of $f(x) = \sqrt[3]{x}$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the horizontal shrink of f.

$h(x) = f(6x)$ — Multiply the input by $1 \div \frac{1}{6} = 6$.

$= \sqrt[3]{6x}$ — Replace x with $6x$ in $f(x)$.

Step 2 Then write a function g that represents the translation of h.

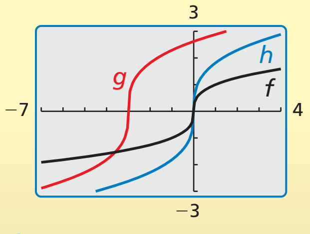
$g(x) = h(x + 3)$ — Subtract −3, or add 3, to the input.

$= \sqrt[3]{6(x + 3)}$ — Replace x with $x + 3$ in $h(x)$.

$= \sqrt[3]{6x + 18}$ — Distributive Property

▶ The transformed function is $g(x) = \sqrt[3]{6x + 18}$.

Check



Monitoring Progress

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3. **WHAT IF?** In Example 3, the function $N(d) = 2.4 \cdot E(d)$ approximates the number of seconds it takes a dropped object to fall d feet on the Moon. Write a rule for N. How long does it take a dropped object to fall 25 feet on the Moon?
4. In Example 4, is the transformed function the same when you perform the translation followed by the horizontal shrink? Explain your reasoning.

Graphing Parabolas and Circles

To graph parabolas and circles using a graphing calculator, first solve their equations for y to obtain radical functions. Then graph the functions.

EXAMPLE 5 Graphing a Parabola (Horizontal Axis of Symmetry)

Use a graphing calculator to graph $\frac{1}{2}y^2 = x$. Identify the vertex and the direction that the parabola opens.

SOLUTION

Step 1 Solve for y.

$\frac{1}{2}y^2 = x$ Write the original equation.

$y^2 = 2x$ Multiply each side by 2.

$y = \pm\sqrt{2x}$ Take square root of each side.

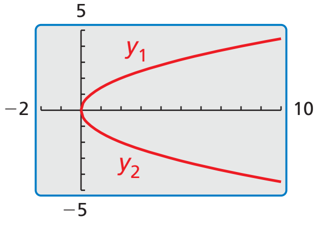
STUDY TIP

Notice y_1 is a function and y_2 is a function, but $\frac{1}{2}y^2 = x$ is not a function.

Step 2 Graph both radical functions.

$$y_1 = \sqrt{2x}$$

$$y_2 = -\sqrt{2x}$$



The vertex is (0, 0) and the parabola opens right.

EXAMPLE 6 Graphing a Circle (Center at the Origin)

Use a graphing calculator to graph $x^2 + y^2 = 16$. Identify the radius and the intercepts.

SOLUTION

Step 1 Solve for y.

$x^2 + y^2 = 16$ Write the original equation.

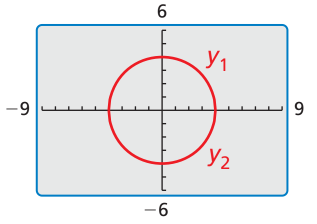
$y^2 = 16 - x^2$ Subtract x^2 from each side.

$y = \pm\sqrt{16 - x^2}$ Take square root of each side.

Step 2 Graph both radical functions using a square viewing window.

$$y_1 = \sqrt{16 - x^2}$$

$$y_2 = -\sqrt{16 - x^2}$$



The radius is 4 units. The x-intercepts are ± 4. The y-intercepts are also ± 4.

Monitoring Progress

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5. Use a graphing calculator to graph $-4y^2 = x + 1$. Identify the vertex and the direction that the parabola opens.

6. Use a graphing calculator to graph $x^2 + y^2 = 25$. Identify the radius and the intercepts.

5.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

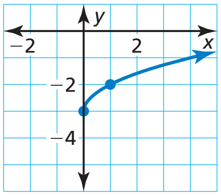
1. **COMPLETE THE SENTENCE** Square root functions and cube root functions are examples of __________ functions.

2. **COMPLETE THE SENTENCE** When graphing $y = a\sqrt[3]{x-h} + k$, translate the graph of $y = a\sqrt[3]{x}$ h units __________ and k units __________.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, match the function with its graph.

3. $f(x) = \sqrt{x+3}$
4. $h(x) = \sqrt{x} + 3$
5. $f(x) = \sqrt{x-3}$
6. $g(x) = \sqrt{x} - 3$
7. $h(x) = \sqrt{x+3} - 3$
8. $f(x) = \sqrt{x-3} + 3$

A.


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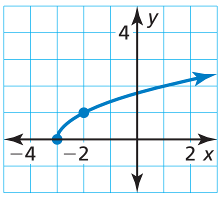

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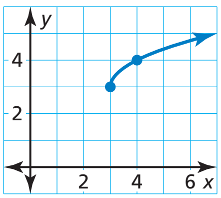

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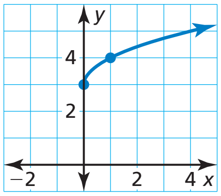

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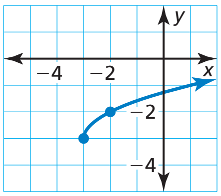

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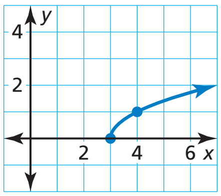


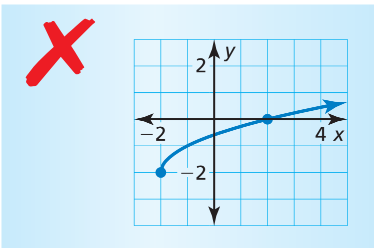
In Exercises 9–18, graph the function. Identify the domain and range of the function. *(See Example 1.)*

9. $h(x) = \sqrt{x} + 4$
10. $g(x) = \sqrt{x} - 5$
11. $g(x) = -\sqrt[3]{2x}$
12. $f(x) = \sqrt[3]{-5x}$
13. $g(x) = \frac{1}{5}\sqrt{x-3}$
14. $f(x) = \frac{1}{2}\sqrt[3]{x+6}$
15. $f(x) = (6x)^{1/2} + 3$
16. $g(x) = -3(x+1)^{1/3}$
17. $h(x) = -\sqrt[4]{x}$
18. $h(x) = \sqrt[5]{2x}$

In Exercises 19–26, describe the transformation of *f* represented by *g*. Then graph each function. *(See Example 2.)*

19. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x+1} + 8$
20. $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x-1}$
21. $f(x) = \sqrt[3]{x}$, $g(x) = -\sqrt[3]{x} - 1$
22. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x+4} - 5$
23. $f(x) = x^{1/2}$, $g(x) = \frac{1}{4}(-x)^{1/2}$
24. $f(x) = x^{1/3}$, $g(x) = \frac{1}{3}x^{1/3} + 6$
25. $f(x) = \sqrt[4]{x}$, $g(x) = 2\sqrt[4]{x+5} - 4$
26. $f(x) = \sqrt[5]{x}$, $g(x) = \sqrt[5]{-32x} + 3$

27. **ERROR ANALYSIS** Describe and correct the error in graphing $f(x) = \sqrt{x-2} - 2$.



28. **ERROR ANALYSIS** Describe and correct the error in describing the transformation of the parent square root function represented by $g(x) = \sqrt{\frac{1}{2}x} + 3$.

✗ The graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ and a translation 3 units up of the parent square root function.

USING TOOLS In Exercises 29–34, use a graphing calculator to graph the function. Then identify the domain and range of the function.

29. $g(x) = \sqrt{x^2 + x}$

30. $h(x) = \sqrt{x^2 - 2x}$

31. $f(x) = \sqrt[3]{x^2 + x}$

32. $f(x) = \sqrt[3]{3x^2 - x}$

33. $f(x) = \sqrt{2x^2 + x + 1}$

34. $h(x) = \sqrt[3]{\frac{1}{2}x^2 - 3x + 4}$

ABSTRACT REASONING In Exercises 35–38, complete the statement with *sometimes*, *always*, or *never*.

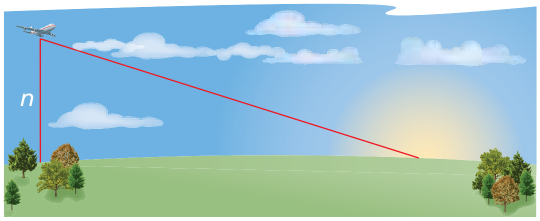
35. The domain of the function $y = a\sqrt{x}$ is ______ $x \geq 0$.

36. The range of the function $y = a\sqrt{x}$ is ______ $y \geq 0$.

37. The domain and range of the function $y = \sqrt[3]{x - h} + k$ are ________ all real numbers.

38. The domain of the function $y = a\sqrt{-x} + k$ is ________ $x \geq 0$.

39. PROBLEM SOLVING The distance (in miles) a pilot can see to the horizon can be approximated by $E(n) = 1.22\sqrt{n}$, where n is the plane's altitude (in feet above sea level) on Earth. The function $M(n) = 0.75E(n)$ approximates the distance a pilot can see to the horizon n feet above the surface of Mars. Write a rule for M. What is the distance a pilot can see to the horizon from an altitude of 10,000 feet above Mars? *(See Example 3.)*



40. MODELING WITH MATHEMATICS The speed (in knots) of sound waves in air can be modeled by

$$v(K) = 643.855\sqrt{\frac{K}{273.15}}$$

where K is the air temperature (in kelvin). The speed (in meters per second) of sound waves in air can be modeled by

$$s(K) = \frac{v(K)}{1.944}.$$

Write a rule for s. What is the speed (in meters per second) of sound waves when the air temperature is 305 kelvin?

In Exercises 41–44, write a rule for g described by the transformations of the graph of f. *(See Example 4.)*

41. Let g be a vertical stretch by a factor of 2, followed by a translation 2 units up of the graph of $f(x) = \sqrt{x} + 3$.

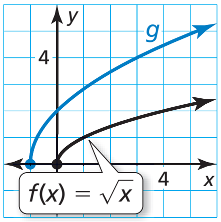
42. Let g be a reflection in the y-axis, followed by a translation 1 unit right of the graph of $f(x) = 2\sqrt[3]{x - 1}$.

43. Let g be a horizontal shrink by a factor of $\frac{2}{3}$, followed by a translation 4 units left of the graph of $f(x) = \sqrt{6x}$.

44. Let g be a translation 1 unit down and 5 units right, followed by a reflection in the x-axis of the graph of $f(x) = -\frac{1}{2}\sqrt[4]{x} + \frac{3}{2}$.

In Exercises 45 and 46, write a rule for g.

45.



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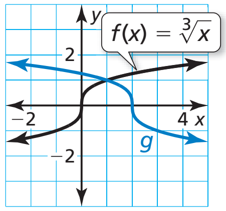


In Exercises 47–50, write a rule for g that represents the indicated transformation of the graph of f.

47. $f(x) = 2\sqrt{x}$, $g(x) = f(x + 3)$

48. $f(x) = \frac{1}{3}\sqrt{x - 1}$, $g(x) = -f(x) + 9$

49. $f(x) = -\sqrt{x^2 - 2}$, $g(x) = -2f(x + 5)$

50. $f(x) = \sqrt[3]{x^2 + 10x}$, $g(x) = \frac{1}{4}f(-x) + 6$

In Exercises 51–56, use a graphing calculator to graph the equation of the parabola. Identify the vertex and the direction that the parabola opens. *(See Example 5.)*

51. $\frac{1}{4}y^2 = x$

52. $3y^2 = x$

53. $-8y^2 + 2 = x$

54. $2y^2 = x - 4$

55. $x + 8 = \frac{1}{5}y^2$

56. $\frac{1}{2}x = y^2 - 4$

In Exercises 57–62, use a graphing calculator to graph the equation of the circle. Identify the radius and the intercepts. *(See Example 6.)*

57. $x^2 + y^2 = 9$

58. $x^2 + y^2 = 4$

59. $1 - y^2 = x^2$

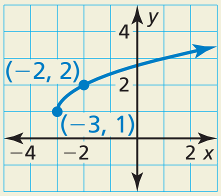
60. $64 - x^2 = y^2$

61. $-y^2 = x^2 - 36$

62. $x^2 = 100 - y^2$

63. MODELING WITH MATHEMATICS The *period* of a pendulum is the time the pendulum takes to complete one back-and-forth swing. The period T (in seconds) can be modeled by the function $T = 1.11\sqrt{\ell}$, where ℓ is the length (in feet) of the pendulum. Graph the function. Estimate the length of a pendulum with a period of 2 seconds. Explain your reasoning.

64. HOW DO YOU SEE IT? Does the graph represent a square root function or a cube root function? Explain. What are the domain and range of the function?



65. PROBLEM SOLVING For a drag race car with a total weight of 3500 pounds, the speed s (in miles per hour) at the end of a race can be modeled by $s = 14.8\sqrt[3]{p}$, where p is the power (in horsepower). Graph the function.

a. Determine the power of a 3500-pound car that reaches a speed of 200 miles per hour.

b. What is the average rate of change in speed as the power changes from 1000 horsepower to 1500 horsepower?

66. THOUGHT PROVOKING The graph of a radical function f passes through the points (3, 1) and (4, 0). Write two different functions that could represent $f(x + 2) + 1$. Explain.

67. MULTIPLE REPRESENTATIONS The terminal velocity v_t (in feet per second) of a skydiver who weighs 140 pounds is given by

$$v_t = 33.7\sqrt{\frac{140}{A}}$$

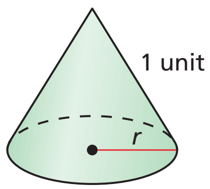
where A is the cross-sectional surface area (in square feet) of the skydiver. The table shows the terminal velocities (in feet per second) for various surface areas (in square feet) of a skydiver who weighs 165 pounds.

Cross-sectional surface area, A	Terminal velocity, v_t
1	432.9
3	249.9
5	193.6
7	163.6

a. Which skydiver has a greater terminal velocity for each value of A given in the table?

b. Describe how the different values of A given in the table relate to the possible positions of the falling skydiver.

68. MATHEMATICAL CONNECTIONS The surface area S of a right circular cone with a slant height of 1 unit is given by $S = \pi r + \pi r^2$, where r is the radius of the cone.



a. Use completing the square to show that

$$r = \frac{1}{\sqrt{\pi}}\sqrt{S + \frac{\pi}{4}} - \frac{1}{2}.$$

b. Graph the equation in part (a) using a graphing calculator. Then find the radius of a right circular cone with a slant height of 1 unit and a surface area of $\frac{3\pi}{4}$ square units.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solutions. *(Skills Review Handbook)*

69. $|3x + 2| = 5$ **70.** $|4x + 9| = -7$ **71.** $|2x - 6| = |x|$ **72.** $|x + 8| = |2x + 2|$

Solve the inequality. *(Section 3.6)*

73. $x^2 + 7x + 12 < 0$ **74.** $x^2 - 10x + 25 \geq 4$ **75.** $2x^2 + 6 > 13x$ **76.** $\frac{1}{8}x^2 + x \leq -2$