5.5 Performing Function Operations

Essential Question

How can you use the graphs of two functions to sketch the graph of an arithmetic combination of the two functions?

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to form other functions. For example, the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, or quotient of f and g.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \qquad \text{sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \qquad \text{difference}$$

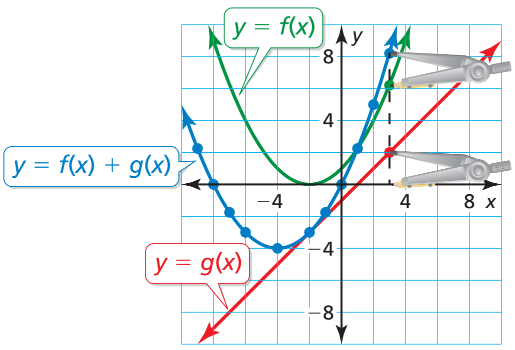
$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \qquad \text{product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1} \qquad \text{quotient}$$

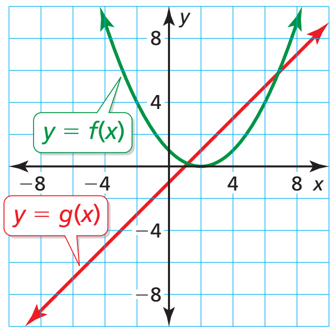
EXPLORATION 1 Graphing the Sum of Two Functions

Work with a partner. Use the graphs of f and g to sketch the graph of $f + g$. Explain your steps.

Sample Choose a point on the graph of g. Use a compass or a ruler to measure its distance above or below the x-axis. If above, add the distance to the y-coordinate of the point with the same x-coordinate on the graph of f. If below, subtract the distance. Plot the new point. Repeat this process for several points. Finally, draw a smooth curve through the new points to obtain the graph of $f + g$.



a.



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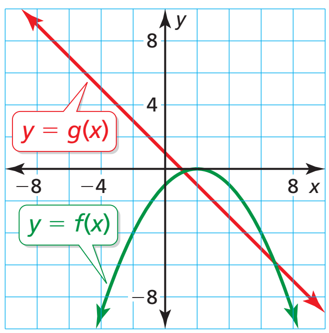


Communicate Your Answer

2. How can you use the graphs of two functions to sketch the graph of an arithmetic combination of the two functions?
3. Check your answers in Exploration 1 by writing equations for f and g, adding the functions, and graphing the sum.

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to check your answers to problems using a different method and continually ask yourself, "Does this make sense?"

5.5 Lesson

Core Vocabulary

Previous
domain
scientific notation

What You Will Learn

- Add, subtract, multiply, and divide functions.

Operations on Functions

You have learned how to add, subtract, multiply, and divide polynomial expressions. These operations can also be defined for functions.

Core Concept

Operations on Functions

Let f and g be any two functions. A new function can be defined by performing any of the four basic operations on f and g.

Operation	Definition	Example: $f(x) = 5x$, $g(x) = x + 2$
Addition	$(f + g)(x) = f(x) + g(x)$	$(f + g)(x) = 5x + (x + 2) = 6x + 2$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$(f - g)(x) = 5x - (x + 2) = 4x - 2$
Multiplication	$(fg)(x) = f(x) \cdot g(x)$	$(fg)(x) = 5x(x + 2) = 5x^2 + 10x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x) = \frac{5x}{x + 2}$

The domains of the sum, difference, product, and quotient functions consist of the x-values that are in the domains of both f and g. Additionally, the domain of the quotient does not include x-values for which $g(x) = 0$.

EXAMPLE 1 Adding Two Functions

Let $f(x) = 3\sqrt{x}$ and $g(x) = -10\sqrt{x}$. Find $(f + g)(x)$ and state the domain. Then evaluate the sum when $x = 4$.

SOLUTION

$$(f + g)(x) = f(x) + g(x) = 3\sqrt{x} + (-10\sqrt{x}) = (3 - 10)\sqrt{x} = -7\sqrt{x}$$

The functions f and g each have the same domain: all nonnegative real numbers. So, the domain of $f + g$ also consists of all nonnegative real numbers. To evaluate $f + g$ when $x = 4$, you can use several methods. Here are two:

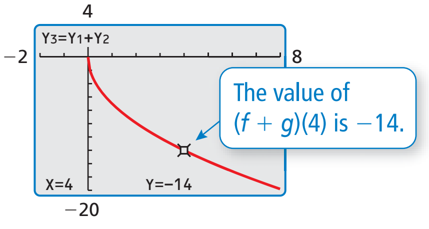
Method 1 Use an algebraic approach.

When $x = 4$, the value of the sum is

$$(f + g)(4) = -7\sqrt{4} = -14.$$

Method 2 Use a graphical approach.

Enter the functions $y_1 = 3\sqrt{x}$, $y_2 = -10\sqrt{x}$, and $y_3 = y_1 + y_2$ in a graphing calculator. Then graph y_3, the sum of the two functions. Use the *trace* feature to find the value of $f + g$ when $x = 4$. From the graph, $(f + g)(4) = -14$.



EXAMPLE 2 Subtracting Two Functions

Let $f(x) = 3x^3 - 2x^2 + 5$ and $g(x) = x^3 - 3x^2 + 4x - 2$. Find $(f - g)(x)$ and state the domain. Then evaluate the difference when $x = -2$.

SOLUTION

$$(f - g)(x) = f(x) - g(x) = 3x^3 - 2x^2 + 5 - (x^3 - 3x^2 + 4x - 2) = 2x^3 + x^2 - 4x + 7$$

The functions f and g each have the same domain: all real numbers. So, the domain of $f - g$ also consists of all real numbers. When $x = -2$, the value of the difference is

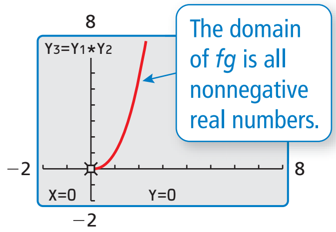
$$(f - g)(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 7 = 3.$$

EXAMPLE 3 Multiplying Two Functions

Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find $(fg)(x)$ and state the domain. Then evaluate the product when $x = 9$.

SOLUTION

$$(fg)(x) = f(x) \cdot g(x) = x^2(\sqrt{x}) = x^2(x^{1/2}) = x^{(2+1/2)} = x^{5/2}$$



The domain of f consists of all real numbers, and the domain of g consists of all nonnegative real numbers. So, the domain of fg consists of all nonnegative real numbers. To confirm this, enter the functions $y_1 = x^2$, $y_2 = \sqrt{x}$, and $y_3 = y_1 \cdot y_2$ in a graphing calculator. Then graph y_3, the product of the two functions. It appears from the graph that the domain of fg consists of all nonnegative real numbers. When $x = 9$, the value of the product is

$$(fg)(9) = 9^{5/2} = (9^{1/2})^5 = 3^5 = 243.$$

EXAMPLE 4 Dividing Two Functions

Let $f(x) = 6x$ and $g(x) = x^{3/4}$. Find $\left(\frac{f}{g}\right)(x)$ and state the domain. Then evaluate the quotient when $x = 16$.

SOLUTION

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{6x}{x^{3/4}} = 6x^{(1-3/4)} = 6x^{1/4}$$

The domain of f consists of all real numbers, and the domain of g consists of all nonnegative real numbers. Because $g(0) = 0$, the domain of $\frac{f}{g}$ is restricted to all *positive* real numbers. When $x = 16$, the value of the quotient is

$$\left(\frac{f}{g}\right)(16) = 6(16)^{1/4} = 6(2^4)^{1/4} = 12.$$

ANOTHER WAY

In Example 4, you can also evaluate $\left(\frac{f}{g}\right)(16)$ as

$$\begin{aligned}\left(\frac{f}{g}\right)(16) &= \frac{f(16)}{g(16)}\\ &= \frac{6(16)}{(16)^{3/4}}\\ &= \frac{96}{8}\\ &= 12.\end{aligned}$$

Monitoring Progress

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1. Let $f(x) = -2x^{2/3}$ and $g(x) = 7x^{2/3}$. Find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $(f + g)(8)$ and $(f - g)(8)$.

2. Let $f(x) = 3x$ and $g(x) = x^{1/5}$. Find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate $(fg)(32)$ and $\left(\frac{f}{g}\right)(32)$.

EXAMPLE 5 Performing Function Operations Using Technology

Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{9 - x^2}$. Use a graphing calculator to evaluate $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ when $x = 2$. Round your answers to two decimal places.

SOLUTION

Enter the functions $y_1 = \sqrt{x}$ and $y_2 = \sqrt{9 - x^2}$ in a graphing calculator. On the home screen, enter $y_1(2) + y_2(2)$. The first entry on the screen shows that $y_1(2) + y_2(2) \approx 3.65$, so $(f + g)(2) \approx 3.65$. Enter the other function operations as shown. Here are the results of the other function operations rounded to two decimal places:

```
Y1(2)+Y2(2)
          3.65028154
Y1(2)-Y2(2)
        -.8218544151
Y1(2)*Y2(2)
          3.16227766
Y1(2)/Y2(2)
          .632455532
```

$$(f - g)(2) \approx -0.82 \qquad (fg)(2) \approx 3.16 \qquad \left(\frac{f}{g}\right)(2) \approx 0.63$$

EXAMPLE 6 Solving a Real-Life Problem

For a white rhino, heart rate r (in beats per minute) and life span s (in minutes) are related to body mass m (in kilograms) by the functions

$$r(m) = 241m^{-0.25}$$

and

$$s(m) = (6 \times 10^6)m^{0.2}.$$

a. Find $(rs)(m)$.

b. Explain what $(rs)(m)$ represents.

SOLUTION

a.
$$\begin{aligned}
(rs)(m) &= r(m) \cdot s(m) && \text{Definition of multiplication} \\
&= 241m^{-0.25}[(6 \times 10^6)m^{0.2}] && \text{Write product of } r(m) \text{ and } s(m). \\
&= 241(6 \times 10^6)m^{-0.25+0.2} && \text{Product of Powers Property} \\
&= (1446 \times 10^6)m^{-0.05} && \text{Simplify.} \\
&= (1.446 \times 10^9)m^{-0.05} && \text{Use scientific notation.}
\end{aligned}$$

b. Multiplying heart rate by life span gives the total number of heartbeats over the lifetime of a white rhino with body mass m.

Monitoring Progress

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3. Let $f(x) = 8x$ and $g(x) = 2x^{5/6}$. Use a graphing calculator to evaluate $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ when $x = 5$. Round your answers to two decimal places.

4. In Example 5, explain why you can evaluate $(f + g)(3)$, $(f - g)(3)$, and $(fg)(3)$ but not $\left(\frac{f}{g}\right)(3)$.

5. Use the answer in Example 6(a) to find the total number of heartbeats over the lifetime of a white rhino when its body mass is 1.7×10^5 kilograms.

5.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Let f and g be any two functions. Describe how you can use f, g, and the four basic operations to create new functions.

2. **WRITING** What x-values are not included in the domain of the quotient of two functions?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $f + g$ and $f - g$ for the given value of x. *(See Examples 1 and 2.)*

3. $f(x) = -5\sqrt[4]{x}$, $g(x) = 19\sqrt[4]{x}$; $x = 16$

4. $f(x) = \sqrt[3]{2x}$, $g(x) = -11\sqrt[3]{2x}$; $x = -4$

5. $f(x) = 6x - 4x^2 - 7x^3$, $g(x) = 9x^2 - 5x$; $x = -1$

6. $f(x) = 11x + 2x^2$, $g(x) = -7x - 3x^2 + 4$; $x = 2$

In Exercises 7–12, find $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and state the domain of each. Then evaluate fg and $\frac{f}{g}$ for the given value of x. *(See Examples 3 and 4.)*

7. $f(x) = 2x^3$, $g(x) = \sqrt[3]{x}$; $x = -27$

8. $f(x) = x^4$, $g(x) = 3\sqrt{x}$; $x = 4$

9. $f(x) = 4x$, $g(x) = 9x^{1/2}$; $x = 9$

10. $f(x) = 11x^3$, $g(x) = 7x^{7/3}$; $x = -8$

11. $f(x) = 7x^{3/2}$, $g(x) = -14x^{1/3}$; $x = 64$

12. $f(x) = 4x^{5/4}$, $g(x) = 2x^{1/2}$; $x = 16$

USING TOOLS **In Exercises 13–16, use a graphing calculator to evaluate $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ when $x = 5$. Round your answers to two decimal places.** *(See Example 5.)*

13. $f(x) = 4x^4$; $g(x) = 24x^{1/3}$

14. $f(x) = 7x^{5/3}$; $g(x) = 49x^{2/3}$

15. $f(x) = -2x^{1/3}$; $g(x) = 5x^{1/2}$

16. $f(x) = 4x^{1/2}$; $g(x) = 6x^{3/4}$

ERROR ANALYSIS **In Exercises 17 and 18, describe and correct the error in stating the domain.**

17. ✗ $f(x) = x^{1/2}$ and $g(x) = x^{3/2}$
The domain of fg is all real numbers.

18. ✗ $f(x) = x^3$ and $g(x) = x^2 - 4$
The domain of $\frac{f}{g}$ is all real numbers except $x = 2$.

19. **MODELING WITH MATHEMATICS** From 1990 to 2010, the numbers (in millions) of female F and male M employees from the ages of 16 to 19 in the United States can be modeled by $F(t) = -0.007t^2 + 0.10t + 3.7$ and $M(t) = 0.0001t^3 - 0.009t^2 + 0.11t + 3.7$, where t is the number of years since 1990. *(See Example 6.)*

    a. Find $(F + M)(t)$.

    b. Explain what $(F + M)(t)$ represents.

20. **MODELING WITH MATHEMATICS** From 2005 to 2009, the numbers of cruise ship departures (in thousands) from around the world W and Florida F can be modeled by the equations

$$W(t) = -5.8333t^3 + 17.43t^2 + 509.1t + 11496$$

$$F(t) = 12.5t^3 - 60.29t^2 + 136.6t + 4881$$

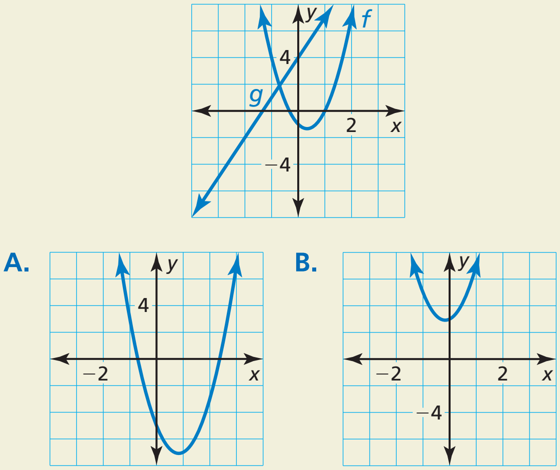
where t is the number of years since 2005.

    a. Find $(W - F)(t)$.

    b. Explain what $(W - F)(t)$ represents.

21. **MAKING AN ARGUMENT** Your friend claims that the addition of functions and the multiplication of functions are commutative. Is your friend correct? Explain your reasoning.

22. HOW DO YOU SEE IT? The graphs of the functions $f(x) = 3x^2 - 2x - 1$ and $g(x) = 3x + 4$ are shown. Which graph represents the function $f + g$? the function $f - g$? Explain your reasoning.

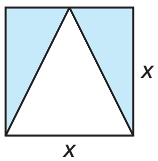


23. REASONING The table shows the outputs of the two functions f and g. Use the table to evaluate $(f + g)(3)$, $(f - g)(1)$, $(fg)(2)$, and $\left(\frac{f}{g}\right)(0)$.

x	0	1	2	3	4
$f(x)$	−2	−4	0	10	26
$g(x)$	−1	−3	−13	−31	−57

24. THOUGHT PROVOKING Is it possible to write two functions whose sum contains radicals, but whose product does not? Justify your answers.

25. MATHEMATICAL CONNECTIONS A triangle is inscribed in a square, as shown. Write and simplify a function r in terms of x that represents the area of the shaded region.



26. REWRITING A FORMULA For a mammal that weighs w grams, the volume b (in milliliters) of air breathed in and the volume d (in milliliters) of "dead space" (the portion of the lungs not filled with air) can be modeled by

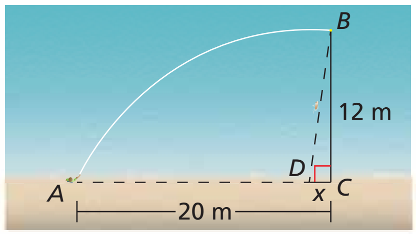
$$b(w) = 0.007w \text{ and } d(w) = 0.002w.$$

The breathing rate r (in breaths per minute) of a mammal that weighs w grams can be modeled by

$$r(w) = \frac{1.1w^{0.734}}{b(w) - d(w)}.$$

Simplify $r(w)$ and calculate the breathing rate for body weights of 6.5 grams, 300 grams, and 70,000 grams.

27. PROBLEM SOLVING A mathematician at a lake throws a tennis ball from point A along the water's edge to point B in the water, as shown. His dog, Elvis, first runs along the beach from point A to point D and then swims to fetch the ball at point B.



a. Elvis runs at a speed of about 6.4 meters per second. Write a function r in terms of x that represents the time he spends running from point A to point D. Elvis swims at a speed of about 0.9 meter per second. Write a function s in terms of x that represents the time he spends swimming from point D to point B.

b. Write a function t in terms of x that represents the total time Elvis spends traveling from point A to point D to point B.

c. Use a graphing calculator to graph t. Find the value of x that minimizes t. Explain the meaning of this value.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the literal equation for n. *(Skills Review Handbook)*

28. $3xn - 9 = 6y$

29. $5z = 7n + 8nz$

30. $3nb = 5n - 6z$

31. $\dfrac{3 + 4n}{n} = 7b$

Determine whether the relation is a function. Explain. *(Skills Review Handbook)*

32. (3, 4), (4, 6), (1, 4), (2, −1)

33. (−1, 2), (3, 7), (0, 2), (−1, −1)

34. (1, 6), (7, −3), (4, 0), (3, 0)

35. (3, 8), (2, 5), (9, 5), (2, −3)