6.7 Modeling with Exponential and Logarithmic Functions

Essential Question

How can you recognize polynomial, exponential, and logarithmic models?

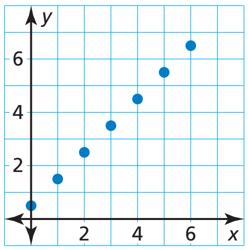
EXPLORATION 1 Recognizing Different Types of Models

Work with a partner. Match each type of model with the appropriate scatter plot. Use a regression program to find a model that fits the scatter plot.

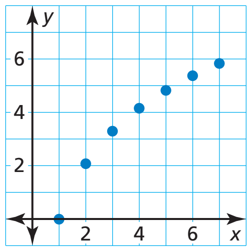
a. linear (positive slope) **b.** linear (negative slope) **c.** quadratic

d. cubic **e.** exponential **f.** logarithmic

A.



B.



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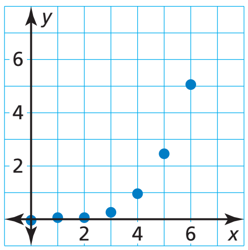

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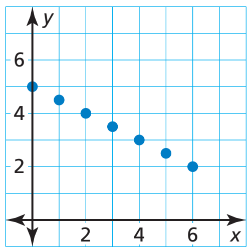

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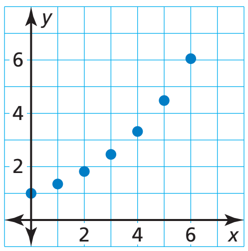
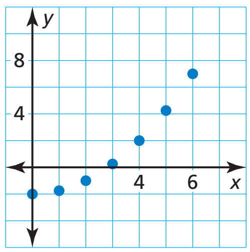


F.



USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

EXPLORATION 2 Exploring Gaussian and Logistic Models

Work with a partner. Two common types of functions that are related to exponential functions are given. Use a graphing calculator to graph each function. Then determine the domain, range, intercept, and asymptote(s) of the function.

a. Gaussian Function: $f(x) = e^{-x^2}$ **b.** Logistic Function: $f(x) = \dfrac{1}{1 + e^{-x}}$

Communicate Your Answer

3. How can you recognize polynomial, exponential, and logarithmic models?

4. Use the Internet or some other reference to find real-life data that can be modeled using one of the types given in Exploration 1. Create a table and a scatter plot of the data. Then use a regression program to find a model that fits the data.

6.7 Lesson

Core Vocabulary

Previous
finite differences
common ratio
point-slope form

What You Will Learn

- Classify data sets.
- Write exponential functions.
- Use technology to find exponential and logarithmic models.

Classifying Data

You have analyzed *finite differences* of data with equally-spaced inputs to determine what type of polynomial function can be used to model the data. For exponential data with equally-spaced inputs, the outputs are multiplied by a constant factor. So, consecutive outputs form a constant ratio.

EXAMPLE 1 Classifying Data Sets

Determine the type of function represented by each table.

a.

x	−2	−1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

b.

x	−2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

SOLUTION

a. The inputs are equally spaced. Look for a pattern in the outputs.

x	−2	−1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

×2 ×2 ×2 ×2 ×2 ×2

▶ As x increases by 1, y is multiplied by 2. So, the common ratio is 2, and the data in the table represent an exponential function.

b. The inputs are equally spaced. The outputs do not have a common ratio. So, analyze the finite differences.

x	−2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

−2 2 6 10 14 18 first differences

4 4 4 4 4 second differences

REMEMBER

First differences of linear functions are constant, second differences of quadratic functions are constant, and so on.

▶ The second differences are constant. So, the data in the table represent a quadratic function.

Monitoring Progress

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Determine the type of function represented by the table. Explain your reasoning.

1.

x	0	10	20	30
y	15	12	9	6

2.

x	0	2	4	6
y	27	9	3	1

Writing Exponential Functions

You know that two points determine a line. Similarly, two points determine an exponential curve.

EXAMPLE 2 Writing an Exponential Function Using Two Points

Write an exponential function $y = ab^x$ whose graph passes through (1, 6) and (3, 54).

SOLUTION

Step 1 Substitute the coordinates of the two given points into $y = ab^x$.

$6 = ab^1$ Equation 1: Substitute 6 for *y* and 1 for *x*.

$54 = ab^3$ Equation 2: Substitute 54 for *y* and 3 for *x*.

Step 2 Solve for a in Equation 1 to obtain $a = \frac{6}{b}$ and substitute this expression for a in Equation 2.

REMEMBER

You know that b must be positive by the definition of an exponential function.

$54 = \left(\frac{6}{b}\right)b^3$ Substitute $\frac{6}{b}$ for *a* in Equation 2.

$54 = 6b^2$ Simplify.

$9 = b^2$ Divide each side by 6.

$3 = b$ Take the positive square root because $b > 0$.

Step 3 Determine that $a = \frac{6}{b} = \frac{6}{3} = 2$.

So, the exponential function is $y = 2(3^x)$.

Data do not always show an *exact* exponential relationship. When the data in a scatter plot show an *approximately* exponential relationship, you can model the data with an exponential function.

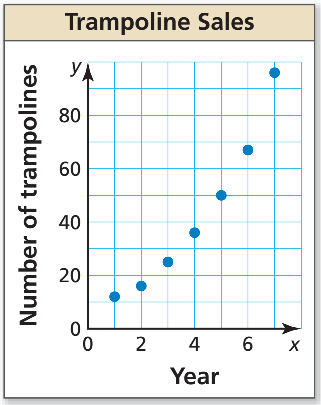
EXAMPLE 3 Finding an Exponential Model

A store sells trampolines. The table shows the numbers y of trampolines sold during the xth year that the store has been open. Write a function that models the data.

Year, x	Number of trampolines, y
1	12
2	16
3	25
4	36
5	50
6	67
7	96

SOLUTION

Step 1 Make a scatter plot of the data. The data appear exponential.



Step 2 Choose any two points to write a model, such as (1, 12) and (4, 36). Substitute the coordinates of these two points into $y = ab^x$.

$12 = ab^1$

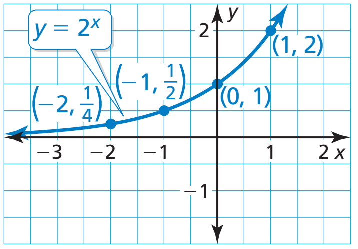
$36 = ab^4$

Solve for a in the first equation to obtain $a = \frac{12}{b}$. Substitute to obtain $b = \sqrt[3]{3} \approx 1.44$ and $a = \frac{12}{\sqrt[3]{3}} \approx 8.32$.

So, an exponential function that models the data is $y = 8.32(1.44)^x$.

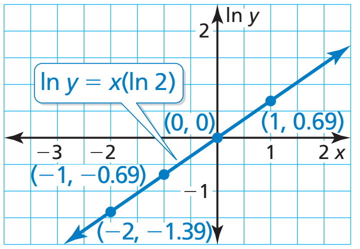
A set of more than two points (x, y) fits an exponential pattern if and only if the set of transformed points $(x, \ln y)$ fits a linear pattern.

Graph of points (x, y)



The graph is an exponential curve.

Graph of points $(x, \ln y)$



The graph is a line.

EXAMPLE 4 Writing a Model Using Transformed Points

Use the data from Example 3. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

SOLUTION

Step 1 Create a table of data pairs $(x, \ln y)$.

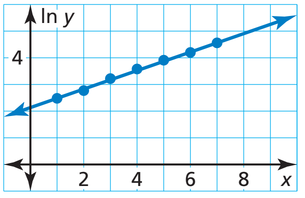
x	1	2	3	4	5	6	7
$\ln y$	2.48	2.77	3.22	3.58	3.91	4.20	4.56

LOOKING FOR STRUCTURE

Because the axes are x and $\ln y$, the point-slope form is rewritten as $\ln y - \ln y_1 = m(x - x_1)$. The slope of the line through (1, 2.48) and (7, 4.56) is

$$\frac{4.56 - 2.48}{7 - 1} \approx 0.35.$$

Step 2 Plot the transformed points as shown. The points lie close to a line, so an exponential model should be a good fit for the original data.



Step 3 Find an exponential model $y = ab^x$ by choosing any two points on the line, such as (1, 2.48) and (7, 4.56). Use these points to write an equation of the line. Then solve for y.

$\ln y - 2.48 = 0.35(x - 1)$ Equation of line

$\ln y = 0.35x + 2.13$ Simplify.

$y = e^{0.35x + 2.13}$ Exponentiate each side using base e.

$y = e^{0.35x}(e^{2.13})$ Use properties of exponents.

$y = 8.41(1.42)^x$ Simplify.

So, an exponential function that models the data is $y = 8.41(1.42)^x$.

Monitoring Progress

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Write an exponential function $y = ab^x$ whose graph passes through the given points.

3. (2, 12), (3, 24)

4. (1, 2), (3, 32)

5. (2, 16), (5, 2)

6. **WHAT IF?** Repeat Examples 3 and 4 using the sales data from another store.

Year, x	1	2	3	4	5	6	7
Number of trampolines, y	15	23	40	52	80	105	140

Using Technology

You can use technology to find best-fit models for exponential and logarithmic data.

EXAMPLE 5 Finding an Exponential Model

Use a graphing calculator to find an exponential model for the data in Example 3. Then use this model and the models in Examples 3 and 4 to predict the number of trampolines sold in the eighth year. Compare the predictions.

SOLUTION

```
ExpReg
 y=a*b^x
 a=8.457377971
 b=1.418848603
 r2=.9972445053
 r=.9986213023
```

Enter the data into a graphing calculator and perform an exponential regression. The model is $y = 8.46(1.42)^x$.

Substitute $x = 8$ into each model to predict the number of trampolines sold in the eighth year.

Example 3: $y = 8.32(1.44)^8 \approx 154$

Example 4: $y = 8.41(1.42)^8 \approx 139$

Regression model: $y = 8.46(1.42)^8 \approx 140$

► The predictions are close for the regression model and the model in Example 4 that used transformed points. These predictions are less than the prediction for the model in Example 3.

EXAMPLE 6 Finding a Logarithmic Model

Weather balloons carry instruments that send back information such as wind speed, temperature, and air pressure.

The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is 1 atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers). Use a graphing calculator to find a logarithmic model of the form $h = a + b \ln p$ that represents the data. Estimate the altitude when the pressure is 0.75 atmosphere.

Air pressure, *p*	1	0.55	0.25	0.12	0.06	0.02
Altitude, *h*	0	5	10	15	20	25

SOLUTION

```
LnReg
 y=a+blnx
 a=.8626578705
 b=-6.447382985
 r2=.9925582287
 r=-.996272166
```

Enter the data into a graphing calculator and perform a logarithmic regression. The model is $h = 0.86 - 6.45 \ln p$.

Substitute $p = 0.75$ into the model to obtain

$h = 0.86 - 6.45 \ln 0.75 \approx 2.7$.

► So, when the air pressure is 0.75 atmosphere, the altitude is about 2.7 kilometers.

Monitoring Progress

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7. Use a graphing calculator to find an exponential model for the data in Monitoring Progress Question 6.

8. Use a graphing calculator to find a logarithmic model of the form $p = a + b \ln h$ for the data in Example 6. Explain why the result is an error message.

6.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Given a set of more than two data pairs (x, y), you can decide whether a(n) __________ function fits the data well by making a scatter plot of the points $(x, \ln y)$.

2. **WRITING** Given a table of values, explain how you can determine whether an exponential function is a good model for a set of data pairs (x, y).

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine the type of function represented by the table. Explain your reasoning. *(See Example 1.)*

3.

x	0	3	6	9	12	15
y	0.25	1	4	16	64	256

4.

x	−4	−3	−2	−1	0	1	2
y	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

5.

x	5	10	15	20	25	30
y	4	3	7	16	30	49

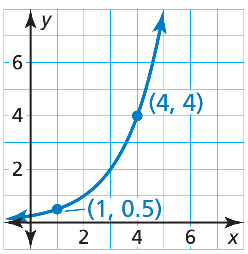
6.

x	−3	1	5	9	13
y	8	−3	−14	−25	−36

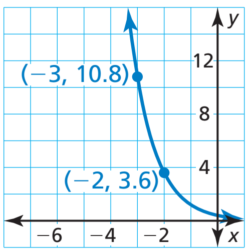
In Exercises 7–16, write an exponential function $y = ab^x$ whose graph passes through the given points. *(See Example 2.)*

7. (1, 3), (2, 12)
8. (2, 24), (3, 144)
9. (3, 1), (5, 4)
10. (3, 27), (5, 243)
11. (1, 2), (3, 50)
12. (1, 40), (3, 640)
13. (−1, 10), (4, 0.31)
14. (2, 6.4), (5, 409.6)

15.



16.



ERROR ANALYSIS **In Exercises 17 and 18, describe and correct the error in determining the type of function represented by the data.**

17.

x	0	1	2	3	4
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

×3 ×3 ×3 ×3

The outputs have a common ratio of 3, so the data represent a linear function.

18.

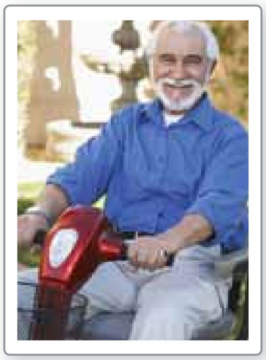
x	−2	−1	1	2	4
y	3	6	12	24	48

×2 ×2 ×2 ×2

The outputs have a common ratio of 2, so the data represent an exponential function.

19. **MODELING WITH MATHEMATICS** A store sells motorized scooters. The table shows the numbers y of scooters sold during the xth year that the store has been open. Write a function that models the data. *(See Example 3.)*

x	y
1	9
2	14
3	19
4	25
5	37
6	53
7	71



20. **MODELING WITH MATHEMATICS** The table shows the numbers y of visits to a website during the xth month. Write a function that models the data. Then use your model to predict the number of visits after 1 year.

x	1	2	3	4	5	6	7
y	22	39	70	126	227	408	735

In Exercises 21–24, determine whether the data show an exponential relationship. Then write a function that models the data.

21.

x	1	6	11	16	21
y	12	28	76	190	450

22.

x	−3	−1	1	3	5
y	2	7	24	68	194

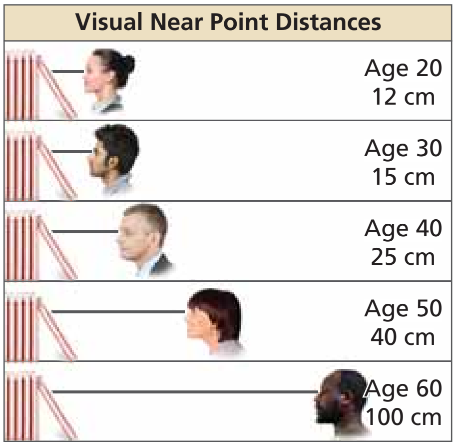
23.

x	0	10	20	30	40	50	60
y	66	58	48	42	31	26	21

24.

x	−20	−13	−6	1	8	15
y	25	19	14	11	8	6

25. **MODELING WITH MATHEMATICS** Your visual near point is the closest point at which your eyes can see an object distinctly. The diagram shows the near point y (in centimeters) at age x (in years). Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data. *(See Example 4.)*



26. **MODELING WITH MATHEMATICS** Use the data from Exercise 19. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

In Exercises 27–30, create a scatter plot of the points $(x, \ln y)$ to determine whether an exponential model fits the data. If so, find an exponential model for the data.

27.

x	1	2	3	4	5
y	18	36	72	144	288

28.

x	1	4	7	10	13
y	3.3	10.1	30.6	92.7	280.9

29.

x	−13	−6	1	8	15
y	9.8	12.2	15.2	19	23.8

30.

x	−8	−5	−2	1	4
y	1.4	1.67	5.32	6.41	7.97

31. **USING TOOLS** Use a graphing calculator to find an exponential model for the data in Exercise 19. Then use the model to predict the number of motorized scooters sold in the tenth year. *(See Example 5.)*

32. **USING TOOLS** A doctor measures an astronaut's pulse rate y (in beats per minute) at various times x (in minutes) after the astronaut has finished exercising. The results are shown in the table. Use a graphing calculator to find an exponential model for the data. Then use the model to predict the astronaut's pulse rate after 16 minutes.

x	y
0	172
2	132
4	110
6	92
8	84
10	78
12	75

33. USING TOOLS An object at a temperature of 160°C is removed from a furnace and placed in a room at 20°C. The table shows the temperatures d (in degrees Celsius) at selected times t (in hours) after the object was removed from the furnace. Use a graphing calculator to find a logarithmic model of the form $t = a + b \ln d$ that represents the data. Estimate how long it takes for the object to cool to 50°C. *(See Example 6.)*

d	160	90	56	38	29	24
t	0	1	2	3	4	5

34. USING TOOLS The f-stops on a camera control the amount of light that enters the camera. Let s be a measure of the amount of light that strikes the film and let f be the f-stop. The table shows several f-stops on a 35-millimeter camera. Use a graphing calculator to find a logarithmic model of the form $s = a + b \ln f$ that represents the data. Estimate the amount of light that strikes the film when $f = 5.657$.

f	**s**
1.414	1
2.000	2
2.828	3
4.000	4
11.314	7

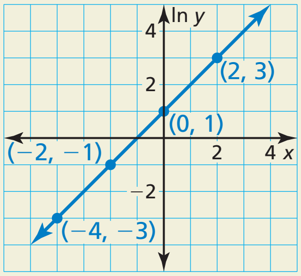
35. DRAWING CONCLUSIONS The table shows the average weight (in kilograms) of an Atlantic cod that is x years old from the Gulf of Maine.

Age, x	1	2	3	4	5
Weight, y	0.751	1.079	1.702	2.198	3.438

a. Show that an exponential model fits the data. Then find an exponential model for the data.

b. By what percent does the weight of an Atlantic cod increase each year in this period of time? Explain.

36. HOW DO YOU SEE IT? The graph shows a set of data points $(x, \ln y)$. Do the data pairs (x, y) fit an exponential pattern? Explain your reasoning.



37. MAKING AN ARGUMENT Your friend says it is possible to find a logarithmic model of the form $d = a + b \ln t$ for the data in Exercise 33. Is your friend correct? Explain.

38. THOUGHT PROVOKING Is it possible to write y as an exponential function of x? Explain your reasoning. (Assume p is positive.)

x	**y**
1	p
2	$2p$
3	$4p$
4	$8p$
5	$16p$

39. CRITICAL THINKING You plant a sunflower seedling in your garden. The height h (in centimeters) of the seedling after t weeks can be modeled by the *logistic function*

$$h(t) = \frac{256}{1 + 13e^{-0.65t}}.$$

a. Find the time it takes the sunflower seedling to reach a height of 200 centimeters.

b. Use a graphing calculator to graph the function. Interpret the meaning of the asymptote in the context of this situation.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Tell whether x and y are in a proportional relationship. Explain your reasoning. *(Skills Review Handbook)*

40. $y = \frac{x}{2}$ **41.** $y = 3x - 12$ **42.** $y = \frac{5}{x}$ **43.** $y = -2x$

Identify the focus, directrix, and axis of symmetry of the parabola. Then graph the equation. *(Section 2.3)*

44. $x = \frac{1}{8}y^2$ **45.** $y = 4x^2$ **46.** $x^2 = 3y$ **47.** $y^2 = \frac{2}{5}x$