7.5 Solving Rational Equations

Essential Question

How can you solve a rational equation?

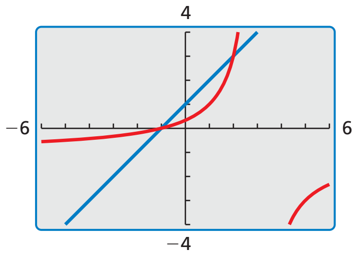
EXPLORATION 1 Solving Rational Equations

Work with a partner. Match each equation with the graph of its related system of equations. Explain your reasoning. Then use the graph to solve the equation.

a. $\dfrac{2}{x-1} = 1$ **b.** $\dfrac{2}{x-2} = 2$ **c.** $\dfrac{-x-1}{x-3} = x + 1$

d. $\dfrac{2}{x-1} = x$ **e.** $\dfrac{1}{x} = \dfrac{-1}{x-2}$ **f.** $\dfrac{1}{x} = x^2$

A.



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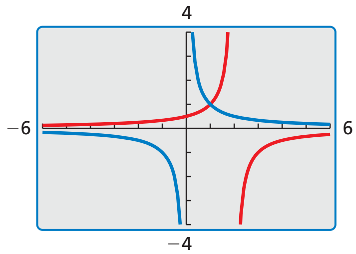

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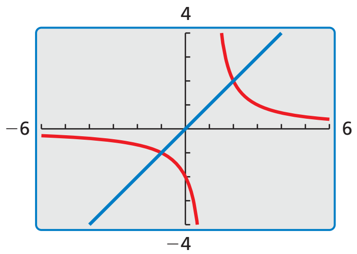
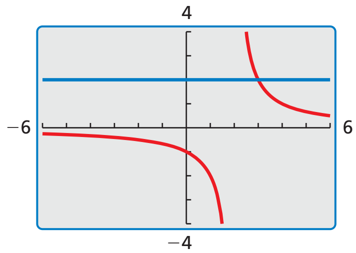


D.



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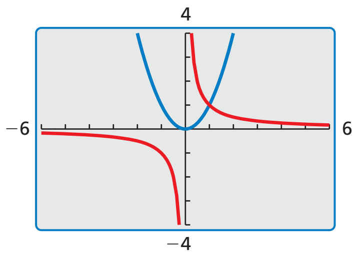
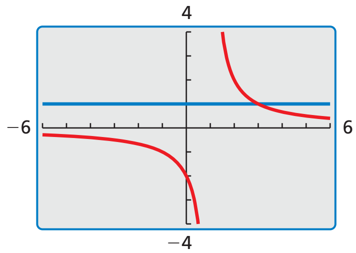


F.



EXPLORATION 2 Solving Rational Equations

Work with a partner. Look back at the equations in Explorations 1(d) and 1(e). Suppose you want a more accurate way to solve the equations than using a graphical approach.

a. Show how you could use a *numerical approach* by creating a table. For instance, you might use a spreadsheet to solve the equations.

b. Show how you could use an *analytical approach*. For instance, you might use the method you used to solve proportions.

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to plan a solution pathway rather than simply jumping into a solution attempt.

Communicate Your Answer

3. How can you solve a rational equation?

4. Use the method in either Exploration 1 or 2 to solve each equation.

a. $\dfrac{x+1}{x-1} = \dfrac{x-1}{x+1}$ **b.** $\dfrac{1}{x+1} = \dfrac{1}{x^2+1}$ **c.** $\dfrac{1}{x^2-1} = \dfrac{1}{x-1}$

7.5 Lesson

Core Vocabulary

cross multiplying, *p. 392*

Previous
proportion
extraneous solution
inverse of a function

What You Will Learn

- Solve rational equations by cross multiplying.
- Solve rational equations by using the least common denominator.
- Use inverses of functions.

Solving by Cross Multiplying

You can use **cross multiplying** to solve a rational equation when each side of the equation is a single rational expression.

EXAMPLE 1 Solving a Rational Equation by Cross Multiplying

Solve $\frac{3}{x+1} = \frac{9}{4x+5}$.

SOLUTION

$\frac{3}{x+1} = \frac{9}{4x+5}$ Write original equation.

$3(4x+5) = 9(x+1)$ Cross multiply.

$12x + 15 = 9x + 9$ Distributive Property

$3x + 15 = 9$ Subtract $9x$ from each side.

$3x = -6$ Subtract 15 from each side.

$x = -2$ Divide each side by 3.

- The solution is $x = -2$. Check this in the original equation.

Check

$$\frac{3}{-2+1} \stackrel{?}{=} \frac{9}{4(-2)+5}$$

$$\frac{3}{-1} \stackrel{?}{=} \frac{9}{-3}$$

$$-3 = -3 \checkmark$$

EXAMPLE 2 Writing and Using a Rational Model

An *alloy* is formed by mixing two or more metals. Sterling silver is an alloy composed of 92.5% silver and 7.5% copper by weight. You have 15 ounces of 800 grade silver, which is 80% silver and 20% copper by weight. How much pure silver should you mix with the 800 grade silver to make sterling silver?

SOLUTION

$$\text{percent of copper in mixture} = \frac{\text{weight of copper in mixture}}{\text{total weight of mixture}}$$

$\frac{7.5}{100} = \frac{(0.2)(15)}{15+x}$ x is the amount of silver added.

$7.5(15+x) = 100(0.2)(15)$ Cross multiply.

$112.5 + 7.5x = 300$ Simplify.

$7.5x = 187.5$ Subtract 112.5 from each side.

$x = 25$ Divide each side by 7.5.

- You should mix 25 ounces of pure silver with the 15 ounces of 800 grade silver.

Monitoring Progress

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Solve the equation by cross multiplying. Check your solution(s).

1. $\frac{3}{5x} = \frac{2}{x-7}$ **2.** $\frac{-4}{x+3} = \frac{5}{x-3}$ **3.** $\frac{1}{2x+5} = \frac{x}{11x+8}$

Solving by Using the Least Common Denominator

When a rational equation is not expressed as a proportion, you can solve it by multiplying each side of the equation by the least common denominator of the rational expressions.

EXAMPLE 3 Solving Rational Equations by Using the LCD

Solve each equation.

a. $\dfrac{5}{x} + \dfrac{7}{4} = -\dfrac{9}{x}$ **b.** $1 - \dfrac{8}{x-5} = \dfrac{3}{x}$

SOLUTION

a.

$$\frac{5}{x} + \frac{7}{4} = -\frac{9}{x}$$ Write original equation.

$$4x\left(\frac{5}{x} + \frac{7}{4}\right) = 4x\left(-\frac{9}{x}\right)$$ Multiply each side by the LCD, $4x$.

$$20 + 7x = -36$$ Simplify.

$$7x = -56$$ Subtract 20 from each side.

$$x = -8$$ Divide each side by 7.

▶ The solution is $x = -8$. Check this in the original equation.

Check

$$\frac{5}{-8} + \frac{7}{4} \stackrel{?}{=} -\frac{9}{-8}$$

$$-\frac{5}{8} + \frac{14}{8} \stackrel{?}{=} \frac{9}{8}$$

$$\frac{9}{8} = \frac{9}{8} \checkmark$$

b.

$$1 - \frac{8}{x-5} = \frac{3}{x}$$ Write original equation.

$$x(x-5)\left(1 - \frac{8}{x-5}\right) = x(x-5) \cdot \frac{3}{x}$$ Multiply each side by the LCD, $x(x-5)$.

$$x(x-5) - 8x = 3(x-5)$$ Simplify.

$$x^2 - 5x - 8x = 3x - 15$$ Distributive Property

$$x^2 - 16x + 15 = 0$$ Write in standard form.

$$(x-1)(x-15) = 0$$ Factor.

$$x = 1 \quad \text{or} \quad x = 15$$ Zero-Product Property

▶ The solutions are $x = 1$ and $x = 15$. Check these in the original equation.

Check

$1 - \dfrac{8}{1-5} \stackrel{?}{=} \dfrac{3}{1}$	Substitute for x.	$1 - \dfrac{8}{15-5} \stackrel{?}{=} \dfrac{3}{15}$
$1 + 2 \stackrel{?}{=} 3$	Simplify.	$1 - \dfrac{4}{5} \stackrel{?}{=} \dfrac{1}{5}$
$3 = 3 \checkmark$		$\dfrac{1}{5} = \dfrac{1}{5} \checkmark$

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Solve the equation by using the LCD. Check your solution(s).

4. $\dfrac{15}{x} + \dfrac{4}{5} = \dfrac{7}{x}$ **5.** $\dfrac{3x}{x+1} - \dfrac{5}{2x} = \dfrac{3}{2x}$ **6.** $\dfrac{4x+1}{x+1} = \dfrac{12}{x^2-1} + 3$

When solving a rational equation, you may obtain solutions that are extraneous. Be sure to check for extraneous solutions by checking your solutions in the *original* equation.

EXAMPLE 4 Solving an Equation with an Extraneous Solution

Solve $\dfrac{6}{x-3} = \dfrac{8x^2}{x^2-9} - \dfrac{4x}{x+3}$.

SOLUTION

Write each denominator in factored form. The LCD is $(x+3)(x-3)$.

$$\frac{6}{x-3} = \frac{8x^2}{(x+3)(x-3)} - \frac{4x}{x+3}$$

$$(x+3)(x-3) \cdot \frac{6}{x-3} = (x+3)(x-3) \cdot \frac{8x^2}{(x+3)(x-3)} - (x+3)(x-3) \cdot \frac{4x}{x+3}$$

$$6(x+3) = 8x^2 - 4x(x-3)$$

$$6x + 18 = 8x^2 - 4x^2 + 12x$$

$$0 = 4x^2 + 6x - 18$$

$$0 = 2x^2 + 3x - 9$$

$$0 = (2x-3)(x+3)$$

$$2x - 3 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = \frac{3}{2} \quad \text{or} \quad x = -3$$

Check

Check $x = \dfrac{3}{2}$:

$$\frac{6}{\frac{3}{2} - 3} \stackrel{?}{=} \frac{8\left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - 9} - \frac{4\left(\frac{3}{2}\right)}{\frac{3}{2} + 3}$$

$$\frac{6}{-\frac{3}{2}} \stackrel{?}{=} \frac{18}{-\frac{27}{4}} - \frac{6}{\frac{9}{2}}$$

$$-4 \stackrel{?}{=} -\frac{8}{3} - \frac{4}{3}$$

$$-4 = -4 \checkmark$$

Check $x = -3$:

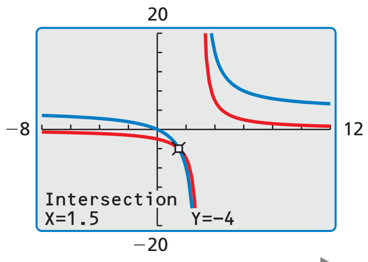
$$\frac{6}{-3-3} \stackrel{?}{=} \frac{8(-3)^2}{(-3)^2 - 9} - \frac{4(-3)}{-3+3}$$

$$\frac{6}{-6} \stackrel{?}{=} \frac{72}{0} - \frac{-12}{0} \quad ✗$$

Division by zero is undefined.

ANOTHER WAY

You can also graph each side of the equation and find the x-value where the graphs intersect.



▶ The apparent solution $x = -3$ is extraneous. So, the only solution is $x = \dfrac{3}{2}$.

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Solve the equation. Check your solution(s).

7. $\dfrac{9}{x-2} + \dfrac{6x}{x+2} = \dfrac{9x^2}{x^2-4}$

8. $\dfrac{7}{x-1} - 5 = \dfrac{6}{x^2-1}$

Using Inverses of Functions

EXAMPLE 5 Finding the Inverse of a Rational Function

Consider the function $f(x) = \dfrac{2}{x+3}$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.

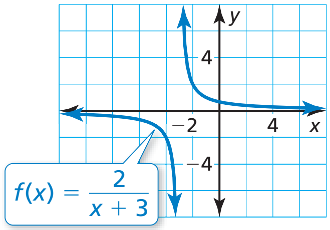
$$y = \frac{2}{x+3} \qquad \text{Set } y \text{ equal to } f(x).$$

$$x = \frac{2}{y+3} \qquad \text{Switch } x \text{ and } y.$$

$$x(y+3) = 2 \qquad \text{Cross multiply.}$$

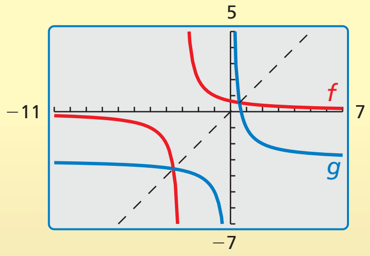
$$y + 3 = \frac{2}{x} \qquad \text{Divide each side by } x.$$

$$y = \frac{2}{x} - 3 \qquad \text{Subtract 3 from each side.}$$



▶ So, the inverse of f is $g(x) = \dfrac{2}{x} - 3$.

Check



EXAMPLE 6 Solving a Real-Life Problem

In Section 7.2 Example 5, you wrote the function $c = \dfrac{50m + 1000}{m}$, which represents the average cost c (in dollars) of making m models using a 3-D printer. Find how many models must be printed for the average cost per model to fall to \$90 by (a) solving an equation, and (b) using the inverse of the function.

REMEMBER

In part (b), the variables are meaningful. Switching them to find the inverse would create confusion. So, solve for m without switching variables.

SOLUTION

a. Substitute 90 for c and solve by cross multiplying.

$$90 = \frac{50m + 1000}{m}$$

$$90m = 50m + 1000$$

$$40m = 1000$$

$$m = 25$$

b. Solve the equation for m.

$$c = \frac{50m + 1000}{m}$$

$$c = 50 + \frac{1000}{m}$$

$$c - 50 = \frac{1000}{m}$$

$$m = \frac{1000}{c - 50}$$

When $c = 90$, $m = \dfrac{1000}{90 - 50} = 25$.

▶ So, the average cost falls to \$90 per model after 25 models are printed.

Monitoring Progress

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9. Consider the function $f(x) = \dfrac{1}{x} - 2$. Determine whether the inverse of f is a function. Then find the inverse.

10. **WHAT IF?** How do the answers in Example 6 change when $c = \dfrac{50m + 800}{m}$?

7.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** When can you solve a rational equation by cross multiplying? Explain.

2. **WRITING** A student solves the equation $\frac{4}{x-3} = \frac{x}{x-3}$ and obtains the solutions 3 and 4. Are either of these extraneous solutions? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, solve the equation by cross multiplying. Check your solution(s). *(See Example 1.)*

3. $\frac{4}{2x} = \frac{5}{x+6}$

4. $\frac{9}{3x} = \frac{4}{x+2}$

5. $\frac{6}{x-1} = \frac{9}{x+1}$

6. $\frac{8}{3x-2} = \frac{2}{x-1}$

7. $\frac{x}{2x+7} = \frac{x-5}{x-1}$

8. $\frac{-2}{x-1} = \frac{x-8}{x+1}$

9. $\frac{x^2-3}{x+2} = \frac{x-3}{2}$

10. $\frac{-1}{x-3} = \frac{x-4}{x^2-27}$

11. **USING EQUATIONS** So far in your volleyball practice, you have put into play 37 of the 44 serves you have attempted. Solve the equation $\frac{90}{100} = \frac{37+x}{44+x}$ to find the number of consecutive serves you need to put into play in order to raise your serve percentage to 90%.

12. **USING EQUATIONS** So far this baseball season, you have 12 hits out of 60 times at-bat. Solve the equation $0.360 = \frac{12+x}{60+x}$ to find the number of consecutive hits you need to raise your batting average to 0.360.

13. **MODELING WITH MATHEMATICS** Brass is an alloy composed of 55% copper and 45% zinc by weight. You have 25 ounces of copper. How many ounces of zinc do you need to make brass? *(See Example 2.)*

14. **MODELING WITH MATHEMATICS** You have 0.2 liter of an acid solution whose acid concentration is 16 moles per liter. You want to dilute the solution with water so that its acid concentration is only 12 moles per liter. Use the given model to determine how many liters of water you should add to the solution.

$$\text{Concentration of new solution} = \frac{\text{Concentration of original solution} \cdot \text{Volume of original solution}}{\text{Volume of original solution} + \text{Volume of water added}}$$

USING STRUCTURE In Exercises 15–18, identify the LCD of the rational expressions in the equation.

15. $\frac{x}{x+3} + \frac{1}{x} = \frac{3}{x}$

16. $\frac{5x}{x-1} - \frac{7}{x} = \frac{9}{x}$

17. $\frac{2}{x+1} + \frac{x}{x+4} = \frac{1}{2}$

18. $\frac{4}{x+9} + \frac{3x}{2x-1} = \frac{10}{3}$

In Exercises 19–30, solve the equation by using the LCD. Check your solution(s). *(See Examples 3 and 4.)*

19. $\frac{3}{2} + \frac{1}{x} = 2$

20. $\frac{2}{3x} + \frac{1}{6} = \frac{4}{3x}$

21. $\frac{x-3}{x-4} + 4 = \frac{3x}{x}$

22. $\frac{2}{x-3} + \frac{1}{x} = \frac{x-1}{x-3}$

23. $\frac{6x}{x+4} + 4 = \frac{2x+2}{x-1}$

24. $\frac{10}{x} + 3 = \frac{x+9}{x-4}$

25. $\frac{18}{x^2-3x} - \frac{6}{x-3} = \frac{5}{x}$

26. $\frac{10}{x^2-2x} + \frac{4}{x} = \frac{5}{x-2}$

27. $\frac{x+1}{x+6} + \frac{1}{x} = \frac{2x+1}{x+6}$

28. $\frac{x+3}{x-3} + \frac{x}{x-5} = \frac{x+5}{x-5}$

29. $\frac{5}{x} - 2 = \frac{2}{x+3}$

30. $\frac{5}{x^2+x-6} = 2 + \frac{x-3}{x-2}$

ERROR ANALYSIS **In Exercises 31 and 32, describe and correct the error in the first step of solving the equation.**

31. ✗

$$\frac{5}{3x} + \frac{2}{x^2} = 1$$

$$3x^3 \cdot \frac{5}{3x} + 3x^3 \cdot \frac{2}{x^2} = 1$$

32. ✗

$$\frac{7x+1}{2x+5} + 4 = \frac{10x-3}{3x}$$

$$(2x+5)3x \cdot \frac{7x+1}{2x+5} + 4 = \frac{10x-3}{3x} \cdot (2x+5)3x$$

33. **PROBLEM SOLVING** You can paint a room in 8 hours. Working together, you and your friend can paint the room in just 5 hours.

a. Let t be the time (in hours) your friend would take to paint the room when working alone. Copy and complete the table.
(*Hint*: (Work done) = (Work rate) × (Time))

	Work rate	Time	Work done
You	$\frac{1 \text{ room}}{8 \text{ hours}}$	5 hours	
Friend		5 hours	

b. Explain what the sum of the expressions represents in the last column. Write and solve an equation to find how long your friend would take to paint the room when working alone.

34. **PROBLEM SOLVING** You can clean a park in 2 hours. Working together, you and your friend can clean the park in just 1.2 hours.

a. Let t be the time (in hours) your friend would take to clean the park when working alone. Copy and complete the table.
(*Hint*: (Work done) = (Work rate) × (Time))

	Work rate	Time	Work done
You	$\frac{1 \text{ park}}{2 \text{ hours}}$	1.2 hours	
Friend		1.2 hours	

b. Explain what the sum of the expressions represents in the last column. Write and solve an equation to find how long your friend would take to clean the park when working alone.

35. **OPEN-ENDED** Give an example of a rational equation that you would solve using cross multiplication and one that you would solve using the LCD. Explain your reasoning.

36. **OPEN-ENDED** Describe a real-life situation that can be modeled by a rational equation. Justify your answer.

In Exercises 37–44, determine whether the inverse of f is a function. Then find the inverse. *(See Example 5.)*

37. $f(x) = \dfrac{2}{x-4}$

38. $f(x) = \dfrac{7}{x+6}$

39. $f(x) = \dfrac{3}{x} - 2$

40. $f(x) = \dfrac{5}{x} - 6$

41. $f(x) = \dfrac{4}{11-2x}$

42. $f(x) = \dfrac{8}{9+5x}$

43. $f(x) = \dfrac{1}{x^2} + 4$

44. $f(x) = \dfrac{1}{x^4} - 7$

45. **PROBLEM SOLVING** The cost of fueling your car for 1 year can be calculated using this equation:

$$\text{Fuel cost for 1 year} = \frac{\text{Miles driven} \cdot \text{Price per gallon of fuel}}{\text{Fuel-efficiency rate}}$$

Last year you drove 9000 miles, paid \$3.24 per gallon of gasoline, and spent a total of \$1389 on gasoline. Find the fuel-efficiency rate of your car by (a) solving an equation, and (b) using the inverse of the function. *(See Example 6.)*

46. **PROBLEM SOLVING** The recommended percent p (in decimal form) of nitrogen (by volume) in the air that a diver breathes is given by $p = \dfrac{105.07}{d+33}$, where d is the depth (in feet) of the diver. Find the depth when the air contains 47% recommended nitrogen by (a) solving an equation, and (b) using the inverse of the function.

USING TOOLS **In Exercises 47–50, use a graphing calculator to solve the equation $f(x) = g(x)$.**

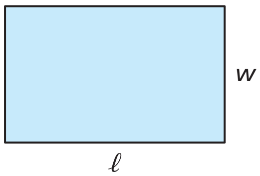
47. $f(x) = \frac{2}{3x}, g(x) = x$

48. $f(x) = -\frac{3}{5x}, g(x) = -x$

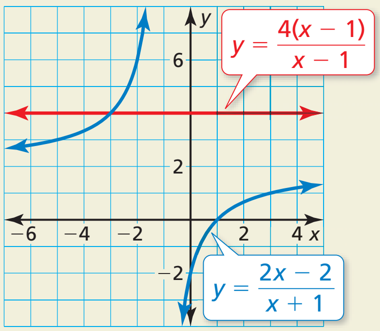
49. $f(x) = \frac{1}{x} + 1, g(x) = x^2$

50. $f(x) = \frac{2}{x} + 1, g(x) = x^2 + 1$

51. **MATHEMATICAL CONNECTIONS** *Golden rectangles* are rectangles for which the ratio of the width w to the length ℓ is equal to the ratio of ℓ to $\ell + w$. The ratio of the length to the width for these rectangles is called the golden ratio. Find the value of the golden ratio using a rectangle with a width of 1 unit.



52. **HOW DO YOU SEE IT?** Use the graph to identify the solution(s) of the rational equation $\frac{4(x-1)}{x-1} = \frac{2x-2}{x+1}$. Explain your reasoning.



USING STRUCTURE **In Exercises 53 and 54, find the inverse of the function. (*Hint:* Try rewriting the function by using either inspection or long division.)**

53. $f(x) = \frac{3x+1}{x-4}$

54. $f(x) = \frac{4x-7}{2x+3}$

55. **ABSTRACT REASONING** Find the inverse of rational functions of the form $y = \frac{ax+b}{cx+d}$. Verify your answer is correct by using it to find the inverses in Exercises 53 and 54.

56. **THOUGHT PROVOKING** Is it possible to write a rational equation that has the following number of solutions? Justify your answers.

a. no solution
b. exactly one solution
c. exactly two solutions
d. infinitely many solutions

57. **CRITICAL THINKING** Let a be a nonzero real number. Tell whether each statement is *always true*, *sometimes true*, or *never true*. Explain your reasoning.

a. For the equation $\frac{1}{x-a} = \frac{x}{x-a}$, $x = a$ is an extraneous solution.

b. The equation $\frac{3}{x-a} = \frac{x}{x-a}$ has exactly one solution.

c. The equation $\frac{1}{x-a} = \frac{2}{x+a} + \frac{2a}{x^2-a^2}$ has no solution.

58. **MAKING AN ARGUMENT** Your friend claims that it is not possible for a rational equation of the form $\frac{x-a}{b} = \frac{x-c}{d}$, where $b \neq 0$ and $d \neq 0$, to have extraneous solutions. Is your friend correct? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Is the domain discrete or continuous? Explain. Graph the function using its domain. *(Skills Review Handbook)*

59. The linear function $y = 0.25x$ represents the amount of money y (in dollars) of x quarters in your pocket. You have a maximum of eight quarters in your pocket.

60. A store sells broccoli for \$2 per pound. The total cost t of the broccoli is a function of the number of pounds p you buy.

Evaluate the function for the given value of x. *(Section 4.1)*

61. $f(x) = x^3 - 2x + 7; x = -2$

62. $g(x) = -2x^4 + 7x^3 + x - 2; x = 3$

63. $h(x) = -x^3 + 3x^2 + 5x; x = 3$

64. $k(x) = -2x^3 - 4x^2 + 12x - 5; x = -5$