8.2 Analyzing Arithmetic Sequences and Series

COMMON CORE

Learning Standards
HSF-IF.A.3
HSF-BF.A.2
HSF-LE.A.2

Essential Question

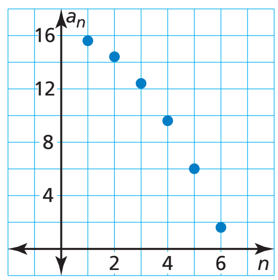
How can you recognize an arithmetic sequence from its graph?

In an **arithmetic sequence**, the difference of consecutive terms, called the *common difference*, is constant. For example, in the arithmetic sequence 1, 4, 7, 10, . . . , the common difference is 3.

EXPLORATION 1 Recognizing Graphs of Arithmetic Sequences

Work with a partner. Determine whether each graph shows an arithmetic sequence. If it does, then write a rule for the nth term of the sequence, and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of an arithmetic sequence?

a.



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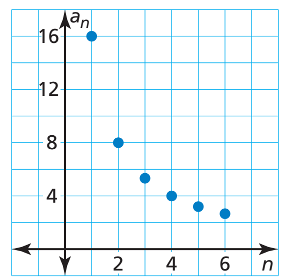

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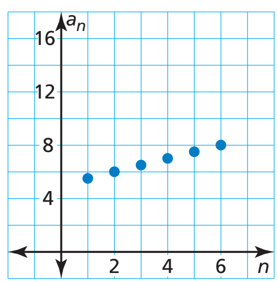

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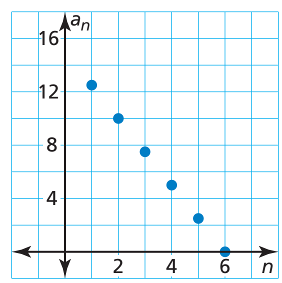

EXPLORATION 2 Finding the Sum of an Arithmetic Sequence

Work with a partner. A teacher of German mathematician Carl Friedrich Gauss (1777–1855) asked him to find the sum of all the whole numbers from 1 through 100. To the astonishment of his teacher, Gauss came up with the answer after only a few moments. Here is what Gauss did:

$$\begin{array}{rcrcrcccr} 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array} \qquad \frac{100 \times 101}{2} = 5050$$

Explain Gauss's thought process. Then write a formula for the sum S_n of the first n terms of an arithmetic sequence. Verify your formula by finding the sums of the first 20 terms of the arithmetic sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

REASONING ABSTRACTLY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

3. How can you recognize an arithmetic sequence from its graph?

4. Find the sum of the terms of each arithmetic sequence.

a. 1, 4, 7, 10, . . . , 301 **b.** 1, 2, 3, 4, . . . , 1000 **c.** 2, 4, 6, 8, . . . , 800

8.2 Lesson

Core Vocabulary

arithmetic sequence, *p. 418*
common difference, *p. 418*
arithmetic series, *p. 420*

Previous
linear function
mean

What You Will Learn

- Identify arithmetic sequences.
- Write rules for arithmetic sequences.
- Find sums of finite arithmetic series.

Identifying Arithmetic Sequences

In an **arithmetic sequence**, the difference of consecutive terms is constant. This constant difference is called the **common difference** and is denoted by d.

EXAMPLE 1 Identifying Arithmetic Sequences

Tell whether each sequence is arithmetic.

a. $-9, -2, 5, 12, 19, \ldots$

b. $23, 15, 9, 5, 3, \ldots$

SOLUTION

Find the differences of consecutive terms.

a. $a_2 - a_1 = -2 - (-9) = 7$

$a_3 - a_2 = 5 - (-2) = 7$

$a_4 - a_3 = 12 - 5 = 7$

$a_5 - a_4 = 19 - 12 = 7$

▶ Each difference is 7, so the sequence is arithmetic.

b. $a_2 - a_1 = 15 - 23 = -8$

$a_3 - a_2 = 9 - 15 = -6$

$a_4 - a_3 = 5 - 9 = -4$

$a_5 - a_4 = 3 - 5 = -2$

▶ The differences are not constant, so the sequence is not arithmetic.

Monitoring Progress

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Tell whether the sequence is arithmetic. Explain your reasoning.

1. $2, 5, 8, 11, 14, \ldots$

2. $15, 9, 3, -3, -9, \ldots$

3. $8, 4, 2, 1, \frac{1}{2}, \ldots$

Writing Rules for Arithmetic Sequences

Core Concept

Rule for an Arithmetic Sequence

Algebra The nth term of an arithmetic sequence with first term a_1 and common difference d is given by:

$$a_n = a_1 + (n - 1)d$$

Example The nth term of an arithmetic sequence with a first term of 3 and a common difference of 2 is given by:

$$a_n = 3 + (n - 1)2, \text{ or } a_n = 2n + 1$$

EXAMPLE 2 Writing a Rule for the *n*th Term

Write a rule for the *n*th term of each sequence. Then find a_{15}.

a. 3, 8, 13, 18, . . . **b.** 55, 47, 39, 31, . . .

SOLUTION

a. The sequence is arithmetic with first term $a_1 = 3$, and common difference $d = 8 - 3 = 5$. So, a rule for the *n*th term is

$a_n = a_1 + (n - 1)d$	Write general rule.
$= 3 + (n - 1)5$	Substitute 3 for a_1 and 5 for d.
$= 5n - 2.$	Simplify.

▶ A rule is $a_n = 5n - 2$, and the 15th term is $a_{15} = 5(15) - 2 = 73$.

b. The sequence is arithmetic with first term $a_1 = 55$, and common difference $d = 47 - 55 = -8$. So, a rule for the *n*th term is

$a_n = a_1 + (n - 1)d$	Write general rule.
$= 55 + (n - 1)(-8)$	Substitute 55 for a_1 and -8 for d.
$= -8n + 63.$	Simplify.

▶ A rule is $a_n = -8n + 63$, and the 15th term is $a_{15} = -8(15) + 63 = -57$.

COMMON ERROR

In the general rule for an arithmetic sequence, note that the common difference d is multiplied by $n - 1$, not n.

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4. Write a rule for the *n*th term of the sequence 7, 11, 15, 19, ... Then find a_{15}.

EXAMPLE 3 Writing a Rule Given a Term and Common Difference

One term of an arithmetic sequence is $a_{19} = -45$. The common difference is $d = -3$. Write a rule for the *n*th term. Then graph the first six terms of the sequence.

SOLUTION

Step 1 Use the general rule to find the first term.

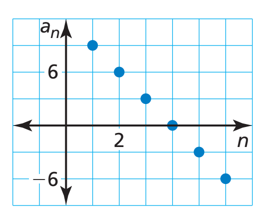
$a_n = a_1 + (n - 1)d$	Write general rule.
$a_{19} = a_1 + (19 - 1)d$	Substitute 19 for n.
$-45 = a_1 + 18(-3)$	Substitute -45 for a_{19} and -3 for d.
$9 = a_1$	Solve for a_1.

Step 2 Write a rule for the *n*th term.

$a_n = a_1 + (n - 1)d$	Write general rule.
$= 9 + (n - 1)(-3)$	Substitute 9 for a_1 and -3 for d.
$= -3n + 12$	Simplify.

Step 3 Use the rule to create a table of values for the sequence. Then plot the points.

n	1	2	3	4	5	6
a_n	9	6	3	0	−3	−6



ANALYZING RELATIONSHIPS

Notice that the points lie on a line. This is true for any arithmetic sequence. So, an arithmetic sequence is a linear function whose domain is a subset of the integers. You can also use function notation to write sequences:

$f(n) = -3n + 12.$

EXAMPLE 4 Writing a Rule Given Two Terms

Two terms of an arithmetic sequence are $a_7 = 17$ and $a_{26} = 93$. Write a rule for the nth term.

SOLUTION

Step 1 Write a system of equations using $a_n = a_1 + (n - 1)d$. Substitute 26 for n to write Equation 1. Substitute 7 for n to write Equation 2.

$a_{26} = a_1 + (26 - 1)d \rightarrow 93 = a_1 + 25d$ Equation 1

$a_7 = a_1 + (7 - 1)d \rightarrow 17 = a_1 + 6d$ Equation 2

Step 2 Solve the system.

$76 = 19d$ Subtract.

$4 = d$ Solve for d.

$93 = a_1 + 25(4)$ Substitute for d in Equation 1.

$-7 = a_1$ Solve for a_1.

Step 3 Write a rule for a_n.

$a_n = a_1 + (n - 1)d$ Write general rule.

$= -7 + (n - 1)4$ Substitute for a_1 and d.

$= 4n - 11$ Simplify.

Check

Use the rule to verify that the 7th term is 17 and the 26th term is 93.

$a_7 = 4(7) - 11 = 17$ ✓

$a_{26} = 4(26) - 11 = 93$ ✓

Monitoring Progress

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Write a rule for the nth term of the sequence. Then graph the first six terms of the sequence.

5. $a_{11} = 50, d = 7$

6. $a_7 = 71, a_{16} = 26$

Finding Sums of Finite Arithmetic Series

The expression formed by adding the terms of an arithmetic sequence is called an **arithmetic series**. The sum of the first n terms of an arithmetic series is denoted by S_n. To find a rule for S_n, you can write S_n in two different ways and add the results.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1 \\ \hline 2S_n &= \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)}_{(a_1 + a_n) \text{ is added } n \text{ times.}} \end{aligned}$$

You can conclude that $2S_n = n(a_1 + a_n)$, which leads to the following result.

Core Concept

The Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is

$$S_n = n\left(\frac{a_1 + a_n}{2}\right).$$

In words, S_n is the mean of the first and nth terms, multiplied by the number of terms.

EXAMPLE 5 Finding the Sum of an Arithmetic Series

Find the sum $\sum_{i=1}^{20} (3i + 7)$.

SOLUTION

Step 1 Find the first and last terms.

$$a_1 = 3(1) + 7 = 10 \qquad \text{Identify first term.}$$

$$a_{20} = 3(20) + 7 = 67 \qquad \text{Identify last term.}$$

Step 2 Find the sum.

$$S_{20} = 20\left(\frac{a_1 + a_{20}}{2}\right) \qquad \text{Write rule for } S_{20}.$$

$$= 20\left(\frac{10 + 67}{2}\right) \qquad \text{Substitute 10 for } a_1 \text{ and 67 for } a_{20}.$$

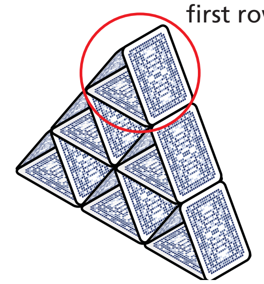
$$= 770 \qquad \text{Simplify.}$$

STUDY TIP

This sum is actually a *partial* sum. You cannot find the complete sum of an infinite arithmetic series because its terms continue indefinitely.

EXAMPLE 6 Solving a Real-Life Problem

You are making a house of cards similar to the one shown.



a. Write a rule for the number of cards in the *n*th row when the top row is row 1.

b. How many cards do you need to make a house of cards with 12 rows?

SOLUTION

a. Starting with the top row, the number of cards in the rows are 3, 6, 9, 12, ... These numbers form an arithmetic sequence with a first term of 3 and a common difference of 3. So, a rule for the sequence is:

$$a_n = a_1 + (n - 1)d \qquad \text{Write general rule.}$$

$$= 3 + (n - 1)(3) \qquad \text{Substitute 3 for } a_1 \text{ and 3 for } d.$$

$$= 3n \qquad \text{Simplify.}$$

b. Find the sum of an arithmetic series with first term $a_1 = 3$ and last term $a_{12} = 3(12) = 36$.

$$S_{12} = 12\left(\frac{a_1 + a_{12}}{2}\right) = 12\left(\frac{3 + 36}{2}\right) = 234$$

▶ So, you need 234 cards to make a house of cards with 12 rows.

Check

Use a graphing calculator to check the sum.

```
sum(seq(3X,X,1,1
2))
                234
```

Monitoring Progress

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Find the sum.

7. $\sum_{i=1}^{10} 9i$

8. $\sum_{k=1}^{12} (7k + 2)$

9. $\sum_{n=1}^{20} (-4n + 6)$

10. **WHAT IF?** In Example 6, how many cards do you need to make a house of cards with eight rows?

8.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The constant difference between consecutive terms of an arithmetic sequence is called the ________________.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

What sequence consists of all the positive odd numbers?
What sequence starts with 1 and has a common difference of 2?
What sequence has an *n*th term of $a_n = 1 + (n - 1)2$?
What sequence has an *n*th term of $a_n = 2n + 1$?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, tell whether the sequence is arithmetic. Explain your reasoning. *(See Example 1.)*

3. $1, -1, -3, -5, -7, \ldots$
4. $12, 6, 0, -6, -12, \ldots$
5. $5, 8, 13, 20, 29, \ldots$
6. $3, 5, 9, 15, 23, \ldots$
7. $36, 18, 9, \frac{9}{2}, \frac{9}{4}, \ldots$
8. $81, 27, 9, 3, 1, \ldots$
9. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \ldots$
10. $\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$

11. **WRITING EQUATIONS** Write a rule for the arithmetic sequence with the given description.
    a. The first term is -3 and each term is 6 less than the previous term.
    b. The first term is 7 and each term is 5 more than the previous term.

12. **WRITING** Compare the terms of an arithmetic sequence when $d > 0$ to when $d < 0$.

In Exercises 13–20, write a rule for the *n*th term of the sequence. Then find a_{20}. *(See Example 2.)*

13. $12, 20, 28, 36, \ldots$
14. $7, 12, 17, 22, \ldots$
15. $51, 48, 45, 42, \ldots$
16. $86, 79, 72, 65, \ldots$
17. $-1, -\frac{1}{3}, \frac{1}{3}, 1, \ldots$
18. $-2, -\frac{5}{4}, -\frac{1}{2}, \frac{1}{4}, \ldots$
19. $2.3, 1.5, 0.7, -0.1, \ldots$
20. $11.7, 10.8, 9.9, 9, \ldots$

ERROR ANALYSIS **In Exercises 21 and 22, describe and correct the error in writing a rule for the *n*th term of the arithmetic sequence 22, 9, −4, −17, −30, ...**

21. ✗
Use $a_1 = 22$ and $d = -13$.
$a_n = a_1 + nd$
$a_n = 22 + n(-13)$
$a_n = 22 - 13n$

22. ✗
The first term is 22 and the common difference is -13.
$a_n = -13 + (n - 1)(22)$
$a_n = -35 + 22n$

In Exercises 23–28, write a rule for the *n*th term of the sequence. Then graph the first six terms of the sequence. *(See Example 3.)*

23. $a_{11} = 43, d = 5$
24. $a_{13} = 42, d = 4$
25. $a_{20} = -27, d = -2$
26. $a_{15} = -35, d = -3$
27. $a_{17} = -5, d = -\frac{1}{2}$
28. $a_{21} = -25, d = -\frac{3}{2}$

29. **USING EQUATIONS** One term of an arithmetic sequence is $a_8 = -13$. The common difference is -8. What is a rule for the *n*th term of the sequence?

    Ⓐ $a_n = 51 + 8n$ Ⓑ $a_n = 35 + 8n$
    Ⓒ $a_n = 51 - 8n$ Ⓓ $a_n = 35 - 8n$

30. **FINDING A PATTERN** One term of an arithmetic sequence is $a_{12} = 43$. The common difference is 6. What is another term of the sequence?

Ⓐ $a_3 = -11$ Ⓑ $a_4 = -53$

Ⓒ $a_5 = 13$ Ⓓ $a_6 = -47$

In Exercises 31–38, write a rule for the *n*th term of the arithmetic sequence. *(See Example 4.)*

31. $a_5 = 41, a_{10} = 96$

32. $a_7 = 58, a_{11} = 94$

33. $a_6 = -8, a_{15} = -62$

34. $a_8 = -15, a_{17} = -78$

35. $a_{18} = -59, a_{21} = -71$

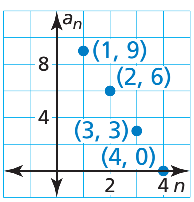
36. $a_{12} = -38, a_{19} = -73$

37. $a_8 = 12, a_{16} = 22$

38. $a_{12} = 9, a_{27} = 15$

WRITING EQUATIONS **In Exercises 39–44, write a rule for the sequence with the given terms.**

39.



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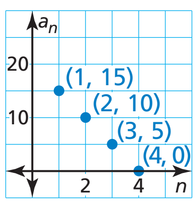

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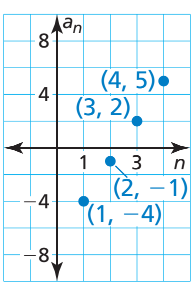

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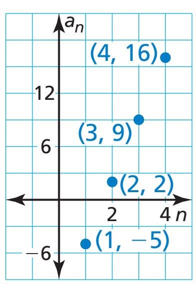


43.

n	4	5	6	7	8
a_n	25	29	33	37	41

44.

n	4	5	6	7	8
a_n	31	39	47	55	63

45. **WRITING** Compare the graph of $a_n = 3n + 1$, where n is a positive integer, with the graph of $f(x) = 3x + 1$, where x is a real number.

46. **DRAWING CONCLUSIONS** Describe how doubling each term in an arithmetic sequence changes the common difference of the sequence. Justify your answer.

In Exercises 47–52, find the sum. *(See Example 5.)*

47. $\sum_{i=1}^{20}(2i - 3)$

48. $\sum_{i=1}^{26}(4i + 7)$

49. $\sum_{i=1}^{33}(6 - 2i)$

50. $\sum_{i=1}^{31}(-3 - 4i)$

51. $\sum_{i=1}^{41}(-2.3 + 0.1i)$

52. $\sum_{i=1}^{39}(-4.1 + 0.4i)$

NUMBER SENSE **In Exercises 53 and 54, find the sum of the arithmetic sequence.**

53. The first 19 terms of the sequence $9, 2, -5, -12, \ldots$.

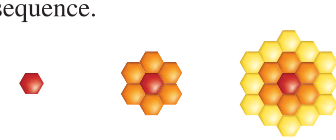
54. The first 22 terms of the sequence $17, 9, 1, -7, \ldots$.

55. **MODELING WITH MATHEMATICS** A marching band is arranged in rows. The first row has three band members, and each row after the first has two more band members than the row before it. *(See Example 6.)*

a. Write a rule for the number of band members in the *n*th row.

b. How many band members are in a formation with seven rows?

56. **MODELING WITH MATHEMATICS** Domestic bees make their honeycomb by starting with a single hexagonal cell, then forming ring after ring of hexagonal cells around the initial cell, as shown. The number of cells in successive rings forms an arithmetic sequence.

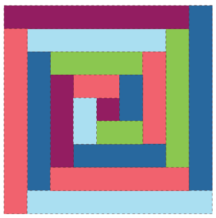


Initial cell 1 ring 2 rings

a. Write a rule for the number of cells in the *n*th ring.

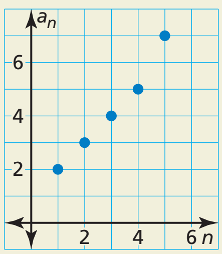
b. How many cells are in the honeycomb after the ninth ring is formed?

57. MATHEMATICAL CONNECTIONS A quilt is made up of strips of cloth, starting with an inner square surrounded by rectangles to form successively larger squares. The inner square and all rectangles have a width of 1 foot. Write an expression using summation notation that gives the sum of the areas of all the strips of cloth used to make the quilt shown. Then evaluate the expression.



58. HOW DO YOU SEE IT? Which graph(s) represents an arithmetic sequence? Explain your reasoning.

a.



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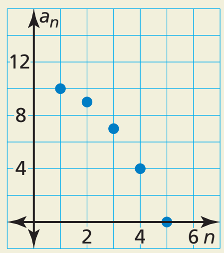

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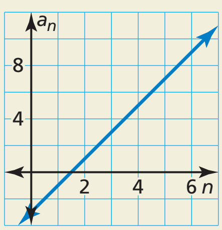

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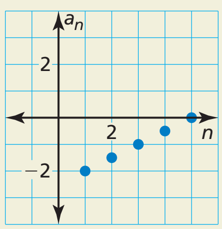

59. MAKING AN ARGUMENT Your friend believes the sum of a series doubles when the common difference of an arithmetic series is doubled and the first term and number of terms in the series remain unchanged. Is your friend correct? Explain your reasoning.

60. THOUGHT PROVOKING In number theory, the *Dirichlet Prime Number Theorem* states that if a and b are relatively prime, then the arithmetic sequence

$$a, a + b, a + 2b, a + 3b, \ldots$$

contains infinitely many prime numbers. Find the first 10 primes in the sequence when $a = 3$ and $b = 4$.

61. REASONING Find the sum of the positive odd integers less than 300. Explain your reasoning.

62. USING EQUATIONS Find the value of n.

a. $\sum_{i=1}^{n}(3i + 5) = 544$ **b.** $\sum_{i=1}^{n}(-4i - 1) = -1127$

c. $\sum_{i=5}^{n}(7 + 12i) = 455$ **d.** $\sum_{i=3}^{n}(-3 - 4i) = -507$

63. ABSTRACT REASONING A theater has n rows of seats, and each row has d more seats than the row in front of it. There are x seats in the last (nth) row and a total of y seats in the entire theater. How many seats are in the front row of the theater? Write your answer in terms of n, x, and y.

64. CRITICAL THINKING The expressions $3 - x$, x, and $1 - 3x$ are the first three terms in an arithmetic sequence. Find the value of x and the next term in the sequence.

65. CRITICAL THINKING One of the major sources of our knowledge of Egyptian mathematics is the Ahmes papyrus, which is a scroll copied in 1650 B.C. by an Egyptian scribe. The following problem is from the Ahmes papyrus.

Divide 10 hekats of barley among 10 men so that the common difference is $\frac{1}{8}$ of a hekat of barley.

Use what you know about arithmetic sequences and series to determine what portion of a hekat each man should receive.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Section 5.2)*

66. $\frac{7}{7^{1/3}}$

67. $\frac{3^{-2}}{3^{-4}}$

68. $\left(\frac{9}{49}\right)^{1/2}$

69. $(5^{1/2} \cdot 5^{1/4})$

Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(Section 6.2)*

70. $y = 2e^x$ **71.** $y = e^{-3x}$ **72.** $y = 3e^{-x}$ **73.** $y = e^{0.25x}$