# 8.3 Analyzing Geometric Sequences and Series

COMMON CORE

Learning Standards
HSA-SSE.B.4
HSF-IF.A.3
HSF-BF.A.2
HSF-LE.A.2

## Essential Question

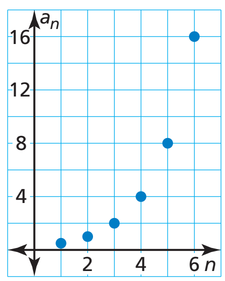
How can you recognize a geometric sequence from its graph?

In a **geometric sequence**, the ratio of any term to the previous term, called the *common ratio*, is constant. For example, in the geometric sequence 1, 2, 4, 8, . . . , the common ratio is 2.

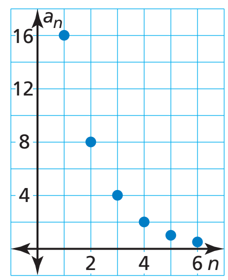
### EXPLORATION 1 Recognizing Graphs of Geometric Sequences

**Work with a partner.** Determine whether each graph shows a geometric sequence. If it does, then write a rule for the $n$th term of the sequence and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of a geometric sequence?

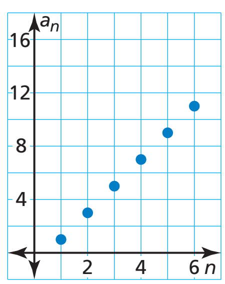
**a.**



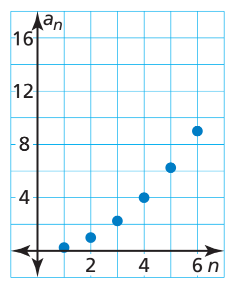

**b.**




**c.**




**d.**




### EXPLORATION 2 Finding the Sum of a Geometric Sequence

**Work with a partner.** You can write the $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$ as

$$a_n = a_1 r^{n-1}.$$

So, you can write the sum $S_n$ of the first $n$ terms of a geometric sequence as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}.$$

Rewrite this formula by finding the difference $S_n - rS_n$ and solve for $S_n$. Then verify your rewritten formula by finding the sums of the first 20 terms of the geometric sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

**LOOKING FOR REGULARITY IN REPEATED REASONING**

To be proficient in math, you need to notice when calculations are repeated, and look both for general methods and for shortcuts.

## Communicate Your Answer

**3.** How can you recognize a geometric sequence from its graph?

**4.** Find the sum of the terms of each geometric sequence.

**a.** 1, 2, 4, 8, . . . , 8192 **b.** 0.1, 0.01, 0.001, 0.0001, . . . , $10^{-10}$

# 8.3 Lesson

## Core Vocabulary

geometric sequence, *p. 426*
common ratio, *p. 426*
geometric series, *p. 428*

***Previous***
exponential function
properties of exponents

## What You Will Learn

- Identify geometric sequences.
- Write rules for geometric sequences.
- Find sums of finite geometric series.

## Identifying Geometric Sequences

In a **geometric sequence**, the ratio of any term to the previous term is constant. This constant ratio is called the **common ratio** and is denoted by $r$.

### EXAMPLE 1 Identifying Geometric Sequences

Tell whether each sequence is geometric.

**a.** 6, 12, 20, 30, 42, . . .

**b.** 256, 64, 16, 4, 1, . . .

**SOLUTION**

Find the ratios of consecutive terms.

**a.** $\frac{a_2}{a_1} = \frac{12}{6} = 2$ $\quad \frac{a_3}{a_2} = \frac{20}{12} = \frac{5}{3}$ $\quad \frac{a_4}{a_3} = \frac{30}{20} = \frac{3}{2}$ $\quad \frac{a_5}{a_4} = \frac{42}{30} = \frac{7}{5}$

▶ The ratios are not constant, so the sequence is not geometric.

**b.** $\frac{a_2}{a_1} = \frac{64}{256} = \frac{1}{4}$ $\quad \frac{a_3}{a_2} = \frac{16}{64} = \frac{1}{4}$ $\quad \frac{a_4}{a_3} = \frac{4}{16} = \frac{1}{4}$ $\quad \frac{a_5}{a_4} = \frac{1}{4}$

▶ Each ratio is $\frac{1}{4}$, so the sequence is geometric.

## Monitoring Progress

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**Tell whether the sequence is geometric. Explain your reasoning.**

**1.** $27, 9, 3, 1, \frac{1}{3}, \ldots$ **2.** 2, 6, 24, 120, 720, . . . **3.** $-1, 2, -4, 8, -16, \ldots$

## Writing Rules for Geometric Sequences

## Core Concept

### Rule for a Geometric Sequence

**Algebra** The $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$ is given by:

$$a_n = a_1 r^{n-1}$$

**Example** The $n$th term of a geometric sequence with a first term of 2 and a common ratio of 3 is given by:

$$a_n = 2(3)^{n-1}$$

## EXAMPLE 2 Writing a Rule for the *n*th Term

Write a rule for the $n$th term of each sequence. Then find $a_8$.

**a.** 5, 15, 45, 135, . . . **b.** 88, −44, 22, −11, . . .

### COMMON ERROR

In the general rule for a geometric sequence, note that the exponent is $n - 1$, not $n$.

### SOLUTION

**a.** The sequence is geometric with first term $a_1 = 5$ and common ratio $r = \frac{15}{5} = 3$. So, a rule for the $n$th term is

$a_n = a_1 r^{n-1}$ Write general rule.

$= 5(3)^{n-1}$. Substitute 5 for $a_1$ and 3 for $r$.

▶ A rule is $a_n = 5(3)^{n-1}$, and the 8th term is $a_8 = 5(3)^{8-1} = 10{,}935$.

**b.** The sequence is geometric with first term $a_1 = 88$ and common ratio $r = \frac{-44}{88} = -\frac{1}{2}$. So, a rule for the $n$th term is

$a_n = a_1 r^{n-1}$ Write general rule.

$= 88\left(-\frac{1}{2}\right)^{n-1}$. Substitute 88 for $a_1$ and $-\frac{1}{2}$ for $r$.

▶ A rule is $a_n = 88\left(-\frac{1}{2}\right)^{n-1}$, and the 8th term is $a_8 = 88\left(-\frac{1}{2}\right)^{8-1} = -\frac{11}{16}$.

## Monitoring Progress

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**4.** Write a rule for the $n$th term of the sequence 3, 15, 75, 375, . . .. Then find $a_9$.

## EXAMPLE 3 Writing a Rule Given a Term and Common Ratio

One term of a geometric sequence is $a_4 = 12$. The common ratio is $r = 2$. Write a rule for the $n$th term. Then graph the first six terms of the sequence.

### SOLUTION

**Step 1** Use the general rule to find the first term.

$a_n = a_1 r^{n-1}$ Write general rule.

$a_4 = a_1 r^{4-1}$ Substitute 4 for $n$.

$12 = a_1(2)^3$ Substitute 12 for $a_4$ and 2 for $r$.

$1.5 = a_1$ Solve for $a_1$.

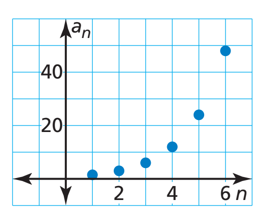
**Step 2** Write a rule for the $n$th term.

$a_n = a_1 r^{n-1}$ Write general rule.

$= 1.5(2)^{n-1}$ Substitute 1.5 for $a_1$ and 2 for $r$.

**Step 3** Use the rule to create a table of values for the sequence. Then plot the points.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a_n$ | 1.5 | 3 | 6 | 12 | 24 | 48 |




### ANALYZING RELATIONSHIPS

Notice that the points lie on an exponential curve because consecutive terms change by equal factors. So, a geometric sequence in which $r > 0$ and $r \neq 1$ is an exponential function whose domain is a subset of the integers.

## EXAMPLE 4 Writing a Rule Given Two Terms

Two terms of a geometric sequence are $a_2 = 12$ and $a_5 = -768$. Write a rule for the $n$th term.

### SOLUTION

**Step 1** Write a system of equations using $a_n = a_1 r^{n-1}$. Substitute 2 for $n$ to write Equation 1. Substitute 5 for $n$ to write Equation 2.

$$a_2 = a_1 r^{2-1} \rightarrow 12 = a_1 r \quad \text{Equation 1}$$

$$a_5 = a_1 r^{5-1} \rightarrow -768 = a_1 r^4 \quad \text{Equation 2}$$

**Step 2** Solve the system.

$$\frac{12}{r} = a_1 \quad \text{Solve Equation 1 for } a_1.$$

$$-768 = \frac{12}{r}(r^4) \quad \text{Substitute for } a_1 \text{ in Equation 2.}$$

$$-768 = 12r^3 \quad \text{Simplify.}$$

$$-4 = r \quad \text{Solve for } r.$$

$$12 = a_1(-4) \quad \text{Substitute for } r \text{ in Equation 1.}$$

$$-3 = a_1 \quad \text{Solve for } a_1.$$

**Step 3** Write a rule for $a_n$.

$$a_n = a_1 r^{n-1} \quad \text{Write general rule.}$$

$$= -3(-4)^{n-1} \quad \text{Substitute for } a_1 \text{ and } r.$$

**Check**

Use the rule to verify that the 2nd term is 12 and the 5th term is $-768$.

$$a_2 = -3(-4)^{2-1}$$
$$= -3(-4) = 12 \checkmark$$
$$a_5 = -3(-4)^{5-1}$$
$$= -3(256) = -768 \checkmark$$

### Monitoring Progress

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**Write a rule for the *n*th term of the sequence. Then graph the first six terms of the sequence.**

**5.** $a_6 = -96, r = -2$

**6.** $a_2 = 12, a_4 = 3$

## Finding Sums of Finite Geometric Series

The expression formed by adding the terms of a geometric sequence is called a **geometric series**. The sum of the first $n$ terms of a geometric series is denoted by $S_n$. You can develop a rule for $S_n$ as follows.

$$\begin{aligned}
S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} \\
-rS_n &= \quad - a_1 r - a_1 r^2 - a_1 r^3 - \cdots - a_1 r^{n-1} - a_1 r^n \\
\hline
S_n - rS_n &= a_1 + 0 + 0 + 0 + \cdots + 0 - a_1 r^n \\
S_n(1 - r) &= a_1(1 - r^n)
\end{aligned}$$

When $r \neq 1$, you can divide each side of this equation by $1 - r$ to obtain the following rule for $S_n$.

### Core Concept

**The Sum of a Finite Geometric Series**

The sum of the first $n$ terms of a geometric series with common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

## EXAMPLE 5 Finding the Sum of a Geometric Series

Find the sum $\sum_{k=1}^{10} 4(3)^{k-1}$.

### SOLUTION

**Step 1** Find the first term and the common ratio.

$a_1 = 4(3)^{1-1} = 4$ Identify first term.

$r = 3$ Identify common ratio.

**Step 2** Find the sum.

$S_{10} = a_1\left(\frac{1 - r^{10}}{1 - r}\right)$ Write rule for $S_{10}$.

$= 4\left(\frac{1 - 3^{10}}{1 - 3}\right)$ Substitute 4 for $a_1$ and 3 for $r$.

$= 118{,}096$ Simplify.

**Check**

Use a graphing calculator to check the sum.

```
sum(seq(4*3^(X-1
),X,1,10))
              118096
```

## EXAMPLE 6 Solving a Real-Life Problem

You can calculate the monthly payment $M$ (in dollars) for a loan using the formula

$$M = \frac{L}{\sum_{k=1}^{t}\left(\frac{1}{1+i}\right)^k}$$

where $L$ is the loan amount (in dollars), $i$ is the monthly interest rate (in decimal form), and $t$ is the term (in months). Calculate the monthly payment on a 5-year loan for \$20,000 with an annual interest rate of 6%.

### SOLUTION

**Step 1** Substitute for $L$, $i$, and $t$. The loan amount is $L = 20{,}000$, the monthly interest rate is $i = \frac{0.06}{12} = 0.005$, and the term is $t = 5(12) = 60$.

$$M = \frac{20{,}000}{\sum_{k=1}^{60}\left(\frac{1}{1+0.005}\right)^k}$$

**Step 2** Notice that the denominator is a geometric series with first term $\frac{1}{1.005}$ and common ratio $\frac{1}{1.005}$. Use a calculator to find the monthly payment.

```
1/1.005→R
        .9950248756
R((1-R^60)/(1-R)
)
        51.72556075
20000/Ans
        386.6560306
```

So, the monthly payment is \$386.66.

**USING TECHNOLOGY**

Storing the value of $\frac{1}{1.005}$ helps minimize mistakes and also assures an accurate answer. Rounding this value to 0.995 results in a monthly payment of \$386.94.

## Monitoring Progress

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**Find the sum.**

**7.** $\sum_{k=1}^{8} 5^{k-1}$

**8.** $\sum_{i=1}^{12} 6(-2)^{i-1}$

**9.** $\sum_{t=1}^{7} -16(0.5)^{t-1}$

**10. WHAT IF?** In Example 6, how does the monthly payment change when the annual interest rate is 5%?

# 8.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

## Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The constant ratio of consecutive terms in a geometric sequence is called the __________.

2. **WRITING** How can you determine whether a sequence is geometric from its graph?

3. **COMPLETE THE SENTENCE** The $n$th term of a geometric sequence has the form $a_n =$ __________.

4. **VOCABULARY** State the rule for the sum of the first $n$ terms of a geometric series.

## Monitoring Progress and Modeling with Mathematics

**In Exercises 5–12, tell whether the sequence is geometric. Explain your reasoning.** *(See Example 1.)*

5. 96, 48, 24, 12, 6, . . .
6. 729, 243, 81, 27, 9, . . .
7. 2, 4, 6, 8, 10, . . .
8. 5, 20, 35, 50, 65, . . .
9. 0.2, 3.2, −12.8, 51.2, −204.8, . . .
10. 0.3, −1.5, 7.5, −37.5, 187.5, . . .
11. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \frac{1}{162}, \ldots$
12. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}, \ldots$

13. **WRITING EQUATIONS** Write a rule for the geometric sequence with the given description.
    a. The first term is −3, and each term is 5 times the previous term.
    b. The first term is 72, and each term is $\frac{1}{3}$ times the previous term.

14. **WRITING** Compare the terms of a geometric sequence when $r > 1$ to when $0 < r < 1$.

**In Exercises 15–22, write a rule for the $n$th term of the sequence. Then find $a_7$.** *(See Example 2.)*

15. 4, 20, 100, 500, . . .
16. 6, 24, 96, 384, . . .
17. 112, 56, 28, 14, . . .
18. 375, 75, 15, 3, . . .
19. $4, 6, 9, \frac{27}{2}, \ldots$
20. $2, \frac{3}{2}, \frac{9}{8}, \frac{27}{32}, \ldots$
21. 1.3, −3.9, 11.7, −35.1, . . .
22. 1.5, −7.5, 37.5, −187.5, . . .

**In Exercises 23–30, write a rule for the $n$th term. Then graph the first six terms of the sequence.** *(See Example 3.)*

23. $a_3 = 4, r = 2$
24. $a_3 = 27, r = 3$
25. $a_2 = 30, r = \frac{1}{2}$
26. $a_2 = 64, r = \frac{1}{4}$
27. $a_4 = -192, r = 4$
28. $a_4 = -500, r = 5$
29. $a_5 = 3, r = -\frac{1}{3}$
30. $a_5 = 1, r = -\frac{1}{5}$

**ERROR ANALYSIS** **In Exercises 31 and 32, describe and correct the error in writing a rule for the $n$th term of the geometric sequence for which $a_2 = 48$ and $r = 6$.**

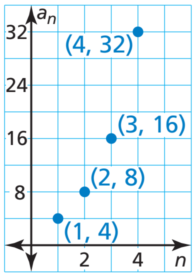
31.
$$a_n = a_1 r^n$$
$$48 = a_1 6^2$$
$$\frac{4}{3} = a_1$$
$$a_n = \frac{4}{3}(6)^n$$

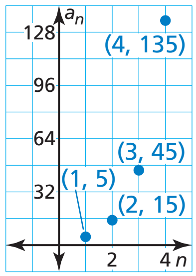
32.
$$a_n = r(a_1)^{n-1}$$
$$48 = 6(a_1)^{2-1}$$
$$8 = a_1$$
$$a_n = 6(8)^{n-1}$$

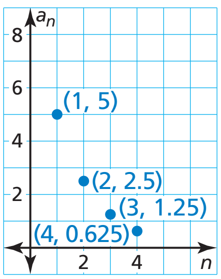
**In Exercises 33–40, write a rule for the $n$th term of the geometric sequence.** *(See Example 4.)*

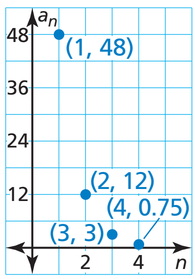
33. $a_2 = 28, a_5 = 1792$
34. $a_1 = 11, a_4 = 88$
35. $a_1 = -6, a_5 = -486$
36. $a_2 = -10, a_6 = -6250$
37. $a_2 = 64, a_4 = 1$
38. $a_1 = 1, a_2 = 49$
39. $a_2 = -72, a_6 = -\frac{1}{18}$
40. $a_2 = -48, a_5 = \frac{3}{4}$

**WRITING EQUATIONS In Exercises 41–46, write a rule for the sequence with the given terms.**

**41.** 


**42.** 


**43.** 


**44.** 


**45.**

| $n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $a_n$ | −12 | 24 | −48 | 96 | −192 |

**46.**

| $n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $a_n$ | −21 | 63 | −189 | 567 | −1701 |

**In Exercises 47–52, find the sum.** *(See Example 5.)*

**47.** $\displaystyle\sum_{i=1}^{9} 6(7)^{i-1}$

**48.** $\displaystyle\sum_{i=1}^{10} 7(4)^{i-1}$

**49.** $\displaystyle\sum_{i=1}^{10} 4\left(\frac{3}{4}\right)^{i-1}$

**50.** $\displaystyle\sum_{i=1}^{8} 5\left(\frac{1}{3}\right)^{i-1}$

**51.** $\displaystyle\sum_{i=0}^{8} 8\left(-\frac{2}{3}\right)^{i}$

**52.** $\displaystyle\sum_{i=0}^{9} 9\left(-\frac{3}{4}\right)^{i}$

**NUMBER SENSE In Exercises 53 and 54, find the sum.**

**53.** The first 8 terms of the geometric sequence −12, −48, −192, −768, . . ..

**54.** The first 9 terms of the geometric sequence −14, −42, −126, −378, . . ..

**55. WRITING** Compare the graph of $a_n = 5(3)^{n-1}$, where $n$ is a positive integer, to the graph of $f(x) = 5 \cdot 3^{x-1}$, where $x$ is a real number.

**56. ABSTRACT REASONING** Use the rule for the sum of a finite geometric series to write each polynomial as a rational expression.

**a.** $1 + x + x^2 + x^3 + x^4$

**b.** $3x + 6x^3 + 12x^5 + 24x^7$

**MODELING WITH MATHEMATICS In Exercises 57 and 58, use the monthly payment formula given in Example 6.**

**57.** You are buying a new car. You take out a 5-year loan for \$15,000. The annual interest rate of the loan is 4%. Calculate the monthly payment. *(See Example 6.)*

**58.** You are buying a new house. You take out a 30-year mortgage for \$200,000. The annual interest rate of the loan is 4.5%. Calculate the monthly payment.


**59. MODELING WITH MATHEMATICS** A regional soccer tournament has 64 participating teams. In the first round of the tournament, 32 games are played. In each successive round, the number of games decreases by a factor of $\frac{1}{2}$.

**a.** Write a rule for the number of games played in the $n$th round. For what values of $n$ does the rule make sense? Explain.

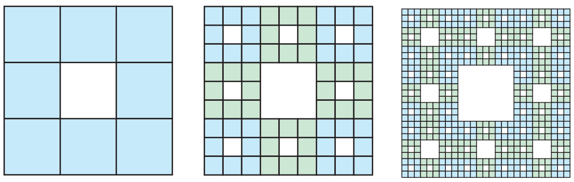
**b.** Find the total number of games played in the regional soccer tournament.

**60. MODELING WITH MATHEMATICS** In a skydiving formation with $R$ rings, each ring after the first has twice as many skydivers as the preceding ring. The formation for $R = 2$ is shown.

**a.** Let $a_n$ be the number of skydivers in the $n$th ring. Write a rule for $a_n$.

**b.** Find the total number of skydivers when there are four rings.

**61. PROBLEM SOLVING** The *Sierpinski carpet* is a fractal created using squares. The process involves removing smaller squares from larger squares. First, divide a large square into nine congruent squares. Then remove the center square. Repeat these steps for each smaller square, as shown below. Assume that each side of the initial square is 1 unit long.



Stage 1 Stage 2 Stage 3

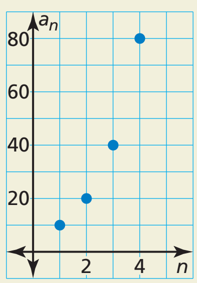
**a.** Let $a_n$ be the total number of squares removed at the $n$th stage. Write a rule for $a_n$. Then find the total number of squares removed through Stage 8.

**b.** Let $b_n$ be the remaining area of the original square after the $n$th stage. Write a rule for $b_n$. Then find the remaining area of the original square after Stage 12.

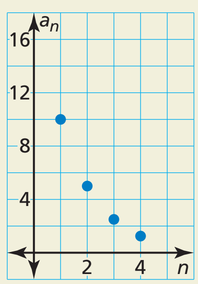
**62. HOW DO YOU SEE IT?** Match each sequence with its graph. Explain your reasoning.

**a.** $a_n = 10\left(\frac{1}{2}\right)^{n-1}$ **b.** $a_n = 10(2)^{n-1}$

**A.**

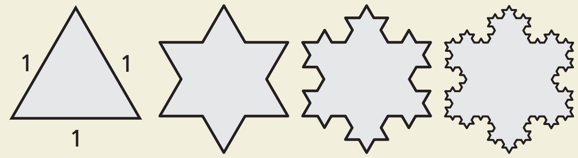



**B.**




**63. CRITICAL THINKING** On January 1, you deposit \$2000 in a retirement account that pays 5% annual interest. You make this deposit each January 1 for the next 30 years. How much money do you have in your account immediately after you make your last deposit?

**64. THOUGHT PROVOKING** The first four iterations of the fractal called the *Koch snowflake* are shown below. Find the perimeter and area of each iteration. Do the perimeters and areas form geometric sequences? Explain your reasoning.




**65. MAKING AN ARGUMENT** You and your friend are comparing two loan options for a \$165,000 house. Loan 1 is a 15-year loan with an annual interest rate of 3%. Loan 2 is a 30-year loan with an annual interest rate of 4%. Your friend claims the total amount repaid over the loan will be less for Loan 2. Is your friend correct? Justify your answer.

**66. CRITICAL THINKING** Let $L$ be the amount of a loan (in dollars), $i$ be the monthly interest rate (in decimal form), $t$ be the term (in months), and $M$ be the monthly payment (in dollars).

**a.** When making monthly payments, you are paying the loan amount plus the interest the loan gathers each month. For a 1-month loan, $t = 1$, the equation for repayment is $L(1 + i) - M = 0$. For a 2-month loan, $t = 2$, the equation is $[L(1 + i) - M](1 + i) - M = 0$. Solve both of these repayment equations for $L$.

**b.** Use the pattern in the equations you solved in part (a) to write a repayment equation for a $t$-month loan. (*Hint*: $L$ is equal to $M$ times a geometric series.) Then solve the equation for $M$.

**c.** Use the rule for the sum of a finite geometric series to show that the formula in part (b) is equivalent to

$$M = L\left(\frac{i}{1 - (1 + i)^{-t}}\right).$$

Use this formula to check your answers in Exercises 57 and 58.

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

**Graph the function. State the domain and range.** *(Section 7.2)*

**67.** $f(x) = \dfrac{1}{x - 3}$

**68.** $g(x) = \dfrac{2}{x} + 3$

**69.** $h(x) = \dfrac{1}{x - 2} + 1$

**70.** $p(x) = \dfrac{3}{x + 1} - 2$