8.5 Using Recursive Rules with Sequences

Essential Question

How can you define a sequence recursively?

A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXPLORATION 1 Evaluating a Recursive Rule

Work with a partner. Use each recursive rule and a spreadsheet to write the first six terms of the sequence. Classify the sequence as arithmetic, geometric, or neither. Explain your reasoning. (The figure shows a partially completed spreadsheet for part (a).)

a. $a_1 = 7,\ a_n = a_{n-1} + 3$

b. $a_1 = 5,\ a_n = a_{n-1} - 2$

c. $a_1 = 1,\ a_n = 2a_{n-1}$

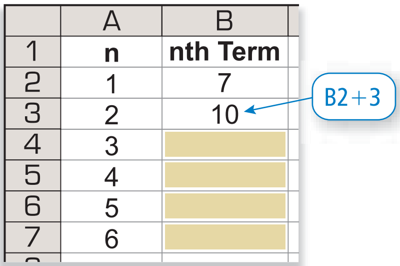
d. $a_1 = 1,\ a_n = \frac{1}{2}(a_{n-1})^2$

e. $a_1 = 3,\ a_n = a_{n-1} + 1$

f. $a_1 = 4,\ a_n = \frac{1}{2}a_{n-1} - 1$

g. $a_1 = 4,\ a_n = \frac{1}{2}a_{n-1}$

h. $a_1 = 4,\ a_2 = 5,\ a_n = a_{n-1} + a_{n-2}$



ATTENDING TO PRECISION

To be proficient in math, you need to communicate precisely to others.

EXPLORATION 2 Writing a Recursive Rule

Work with a partner. Write a recursive rule for the sequence. Explain your reasoning.

a. 3, 6, 9, 12, 15, 18, . . .

b. 18, 14, 10, 6, 2, −2, . . .

c. 3, 6, 12, 24, 48, 96, . . .

d. 128, 64, 32, 16, 8, 4, . . .

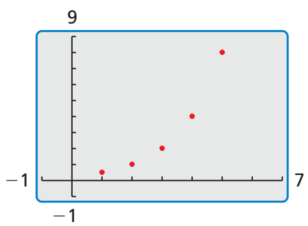
e. 5, 5, 5, 5, 5, 5, . . .

f. 1, 1, 2, 3, 5, 8, . . .

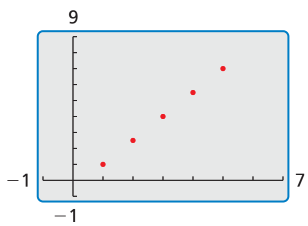
EXPLORATION 3 Writing a Recursive Rule

Work with a partner. Write a recursive rule for the sequence whose graph is shown.

a.



b.



Communicate Your Answer

4. How can you define a sequence recursively?

5. Write a recursive rule that is different from those in Explorations 1–3. Write the first six terms of the sequence. Then graph the sequence and classify it as arithmetic, geometric, or neither.

8.5 Lesson

Core Vocabulary

explicit rule, *p. 442*
recursive rule, *p. 442*

What You Will Learn

- Evaluate recursive rules for sequences.
- Write recursive rules for sequences.
- Translate between recursive and explicit rules for sequences.
- Use recursive rules to solve real-life problems.

Evaluating Recursive Rules

So far in this chapter, you have worked with explicit rules for the nth term of a sequence, such as $a_n = 3n - 2$ and $a_n = 7(0.5)^n$. An **explicit rule** gives a_n as a function of the term's position number n in the sequence.

In this section, you will learn another way to define a sequence—by a *recursive rule*. A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXAMPLE 1 Evaluating Recursive Rules

Write the first six terms of each sequence.

a. $a_0 = 1,\ a_n = a_{n-1} + 4$

b. $f(1) = 1,\ f(n) = 3 \cdot f(n-1)$

SOLUTION

a.

$a_0 = 1$	1st term
$a_1 = a_0 + 4 = 1 + 4 = 5$	2nd term
$a_2 = a_1 + 4 = 5 + 4 = 9$	3rd term
$a_3 = a_2 + 4 = 9 + 4 = 13$	4th term
$a_4 = a_3 + 4 = 13 + 4 = 17$	5th term
$a_5 = a_4 + 4 = 17 + 4 = 21$	6th term

b.

$f(1) = 1$

$f(2) = 3 \cdot f(1) = 3(1) = 3$

$f(3) = 3 \cdot f(2) = 3(3) = 9$

$f(4) = 3 \cdot f(3) = 3(9) = 27$

$f(5) = 3 \cdot f(4) = 3(27) = 81$

$f(6) = 3 \cdot f(5) = 3(81) = 243$

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Write the first six terms of the sequence.

1. $a_1 = 3,\ a_n = a_{n-1} - 7$

2. $a_0 = 162,\ a_n = 0.5a_{n-1}$

3. $f(0) = 1,\ f(n) = f(n-1) + n$

4. $a_1 = 4,\ a_n = 2a_{n-1} - 1$

Writing Recursive Rules

In part (a) of Example 1, the *differences* of consecutive terms of the sequence are constant, so the sequence is arithmetic. In part (b), the *ratios* of consecutive terms are constant, so the sequence is geometric. In general, rules for arithmetic and geometric sequences can be written recursively as follows.

Core Concept

Recursive Equations for Arithmetic and Geometric Sequences

Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Geometric Sequence

$a_n = r \cdot a_{n-1}$, where r is the common ratio

EXAMPLE 2 Writing Recursive Rules

Write a recursive rule for (a) 3, 13, 23, 33, 43, . . . and (b) 16, 40, 100, 250, 625, ...

SOLUTION

Use a table to organize the terms and find the pattern.

a.

n	1	2	3	4	5
a_n	3	13	23	33	43

$+10 \quad +10 \quad +10 \quad +10$

The sequence is arithmetic with first term $a_1 = 3$ and common difference $d = 10$.

$$a_n = a_{n-1} + d \qquad \text{Recursive equation for arithmetic sequence}$$

$$= a_{n-1} + 10 \qquad \text{Substitute 10 for } d.$$

▶ A recursive rule for the sequence is $a_1 = 3$, $a_n = a_{n-1} + 10$.

b.

n	1	2	3	4	5
a_n	16	40	100	250	625

$\times\frac{5}{2} \quad \times\frac{5}{2} \quad \times\frac{5}{2} \quad \times\frac{5}{2}$

The sequence is geometric with first term $a_1 = 16$ and common ratio $r = \frac{5}{2}$.

$$a_n = r \cdot a_{n-1} \qquad \text{Recursive equation for geometric sequence}$$

$$= \frac{5}{2}a_{n-1} \qquad \text{Substitute } \frac{5}{2} \text{ for } r.$$

▶ A recursive rule for the sequence is $a_1 = 16$, $a_n = \frac{5}{2}a_{n-1}$.

COMMON ERROR

A recursive *equation* for a sequence does not include the initial term. To write a recursive *rule* for a sequence, the initial term(s) must be included.

STUDY TIP

The sequence in part (a) of Example 3 is called the *Fibonacci sequence*. The sequence in part (b) lists *factorial numbers*. You will learn more about *factorials* in Chapter 10.

EXAMPLE 3 Writing Recursive Rules

Write a recursive rule for each sequence.

a. 1, 1, 2, 3, 5, . . .

b. 1, 1, 2, 6, 24, . . .

SOLUTION

a. The terms have neither a common difference nor a common ratio. Beginning with the third term in the sequence, each term is the sum of the two previous terms.

▶ A recursive rule for the sequence is $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$.

b. The terms have neither a common difference nor a common ratio. Denote the first term by $a_0 = 1$. Note that $a_1 = 1 = 1 \cdot a_0$, $a_2 = 2 = 2 \cdot a_1$, $a_3 = 6 = 3 \cdot a_2$, and so on.

▶ A recursive rule for the sequence is $a_0 = 1$, $a_n = n \cdot a_{n-1}$.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Write a recursive rule for the sequence.

5. 2, 14, 98, 686, 4802, . . .

6. 19, 13, 7, 1, −5, . . .

7. 11, 22, 33, 44, 55, . . .

8. 1, 2, 2, 4, 8, 32, . . .

Translating Between Recursive and Explicit Rules

EXAMPLE 4 Translating from Explicit Rules to Recursive Rules

Write a recursive rule for (a) $a_n = -6 + 8n$ and (b) $a_n = -3\left(\frac{1}{2}\right)^{n-1}$.

SOLUTION

a. The explicit rule represents an arithmetic sequence with first term $a_1 = -6 + 8(1) = 2$ and common difference $d = 8$.

$a_n = a_{n-1} + d$ — Recursive equation for arithmetic sequence

$a_n = a_{n-1} + 8$ — Substitute 8 for *d*.

▶ A recursive rule for the sequence is $a_1 = 2$, $a_n = a_{n-1} + 8$.

b. The explicit rule represents a geometric sequence with first term $a_1 = -3\left(\frac{1}{2}\right)^0 = -3$ and common ratio $r = \frac{1}{2}$.

$a_n = r \cdot a_{n-1}$ — Recursive equation for geometric sequence

$a_n = \frac{1}{2}a_{n-1}$ — Substitute $\frac{1}{2}$ for *r*.

▶ A recursive rule for the sequence is $a_1 = -3$, $a_n = \frac{1}{2}a_{n-1}$.

EXAMPLE 5 Translating from Recursive Rules to Explicit Rules

Write an explicit rule for each sequence.

a. $a_1 = -5$, $a_n = a_{n-1} - 2$ **b.** $a_1 = 10$, $a_n = 2a_{n-1}$

SOLUTION

a. The recursive rule represents an arithmetic sequence with first term $a_1 = -5$ and common difference $d = -2$.

$a_n = a_1 + (n - 1)d$ — Explicit rule for arithmetic sequence

$a_n = -5 + (n - 1)(-2)$ — Substitute -5 for a_1 and -2 for *d*.

$a_n = -3 - 2n$ — Simplify.

▶ An explicit rule for the sequence is $a_n = -3 - 2n$.

b. The recursive rule represents a geometric sequence with first term $a_1 = 10$ and common ratio $r = 2$.

$a_n = a_1 r^{n-1}$ — Explicit rule for geometric sequence

$a_n = 10(2)^{n-1}$ — Substitute 10 for a_1 and 2 for *r*.

▶ An explicit rule for the sequence is $a_n = 10(2)^{n-1}$.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Write a recursive rule for the sequence.

9. $a_n = 17 - 4n$ **10.** $a_n = 16(3)^{n-1}$

Write an explicit rule for the sequence.

11. $a_1 = -12$, $a_n = a_{n-1} + 16$ **12.** $a_1 = 2$, $a_n = -6a_{n-1}$

Solving Real-Life Problems

EXAMPLE 6 Solving a Real-Life Problem

A lake initially contains 5200 fish. Each year, the population declines 30% due to fishing and other causes, so the lake is restocked with 400 fish.

a. Write a recursive rule for the number a_n of fish at the start of the nth year.

b. Find the number of fish at the start of the fifth year.

c. Describe what happens to the population of fish over time.

SOLUTION

a. Write a recursive rule. The initial value is 5200. Because the population declines 30% each year, 70% of the fish remain in the lake from one year to the next. Also, 400 fish are added each year. Here is a verbal model for the recursive equation.

Fish at start of year n $= 0.7 \cdot$ Fish at start of year $n - 1$ $+$ New fish added

$$a_n = 0.7 \cdot a_{n-1} + 400$$

▶ A recursive rule is $a_1 = 5200$, $a_n = (0.7)a_{n-1} + 400$.

b. Find the number of fish at the start of the fifth year. Enter 5200 (the value of a_1) in a graphing calculator. Then enter the rule

$.7 \times \text{Ans} + 400$

to find a_2. Press the enter button three more times to find $a_5 \approx 2262$.

```
5200
                5200
.7*Ans+400
                4040
                3228
              2659.6
             2261.72
```

▶ There are about 2262 fish in the lake at the start of the fifth year.

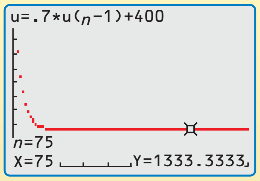
c. Describe what happens to the population of fish over time. Continue pressing enter on the calculator. The screen at the right shows the fish populations for years 44 to 50. Observe that the population of fish approaches 1333.

```
         1333.334178
         1333.333924
         1333.333747
         1333.333623
         1333.333536
         1333.333475
         1333.333433
```

▶ Over time, the population of fish in the lake stabilizes at about 1333 fish.

Check

Set a graphing calculator to *sequence* and *dot* modes. Graph the sequence and use the *trace* feature. From the graph, it appears the sequence approaches 1333.



Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

13. WHAT IF? In Example 6, suppose 75% of the fish remain each year. What happens to the population of fish over time?

EXAMPLE 7 Modeling with Mathematics

REMEMBER

In Section 8.3, you used a formula involving a geometric series to calculate the monthly payment for a similar loan.

You borrow \$150,000 at 6% annual interest compounded monthly for 30 years. The monthly payment is \$899.33.

- Find the balance after the third payment.
- Due to rounding in the calculations, the last payment is often different from the original payment. Find the amount of the last payment.

SOLUTION

1. **Understand the Problem** You are given the conditions of a loan. You are asked to find the balance after the third payment and the amount of the last payment.

2. **Make a Plan** Because the balance after each payment depends on the balance after the previous payment, write a recursive rule that gives the balance after each payment. Then use a spreadsheet to find the balance after each payment, rounded to the nearest cent.

3. **Solve the Problem** Because the monthly interest rate is $\frac{0.06}{12} = 0.005$, the balance increases by a factor of 1.005 each month, and then the payment of \$899.33 is subtracted.

$$\text{Balance after payment} = 1.005 \cdot \text{Balance before payment} - \text{Payment}$$

$$a_n = 1.005 \cdot a_{n-1} - 899.33$$

Use a spreadsheet and the recursive rule to find the balance after the third payment and after the 359th payment.

	A	B	
1	Payment number	Balance after payment	
2	1	149850.67	B2 =Round(1.005*150000−899.33, 2)
3	2	149700.59	B3 =Round(1.005*B2−899.33, 2)
4	3	149549.76	⋮
358	357	2667.38	
359	358	1781.39	
360	359	890.97	B360 =Round(1.005*B359−899.33, 2)

▶ The balance after the third payment is \$149,549.76. The balance after the 359th payment is \$890.97, so the final payment is 1.005(890.97) = \$895.42.

4. **Look Back** By continuing the spreadsheet for the 360th payment using the original monthly payment of \$899.33, the balance is −3.91.

361	360	−3.91	B361 =Round(1.005*B360−899.33, 2)

This shows an overpayment of \$3.91. So, it is reasonable that the last payment is \$899.33 − \$3.91 = \$895.42.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

14. WHAT IF? How do the answers in Example 7 change when the annual interest rate is 7.5% and the monthly payment is \$1048.82?

8.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A recursive _________ tells how the nth term of a sequence is related to one or more preceding terms.

2. **WRITING** Explain the difference between an explicit rule and a recursive rule for a sequence.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, write the first six terms of the sequence. *(See Example 1.)*

3. $a_1 = 1$
$a_n = a_{n-1} + 3$

4. $a_1 = 1$
$a_n = a_{n-1} - 5$

5. $f(0) = 4$
$f(n) = 2f(n-1)$

6. $f(0) = 10$
$f(n) = \frac{1}{2}f(n-1)$

7. $a_1 = 2$
$a_n = (a_{n-1})^2 + 1$

8. $a_1 = 1$
$a_n = (a_{n-1})^2 - 10$

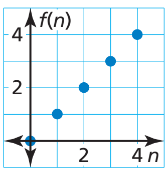
9. $f(0) = 2, f(1) = 4$
$f(n) = f(n-1) - f(n-2)$

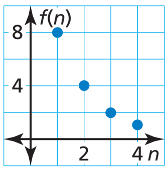
10. $f(1) = 2, f(2) = 3$
$f(n) = f(n-1) \cdot f(n-2)$

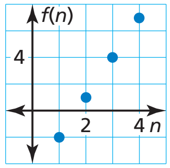
In Exercises 11–22, write a recursive rule for the sequence. *(See Examples 2 and 3.)*

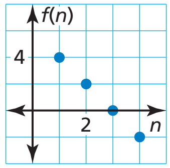
11. $21, 14, 7, 0, -7, \ldots$
12. $54, 43, 32, 21, 10, \ldots$
13. $3, 12, 48, 192, 768, \ldots$
14. $4, -12, 36, -108, \ldots$
15. $44, 11, \frac{11}{4}, \frac{11}{16}, \frac{11}{64}, \ldots$
16. $1, 8, 15, 22, 29, \ldots$
17. $2, 5, 10, 50, 500, \ldots$
18. $3, 5, 15, 75, 1125, \ldots$
19. $1, 4, 5, 9, 14, \ldots$
20. $16, 9, 7, 2, 5, \ldots$
21. $6, 12, 36, 144, 720, \ldots$
22. $-3, -1, 2, 6, 11, \ldots$

In Exercises 23–26, write a recursive rule for the sequence shown in the graph.

23. 

24. 

25. 

26. 

ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in writing a recursive rule for the sequence 5, 2, 3, −1, 4, ...**

27. ✗ Beginning with the third term in the sequence, each term a_n equals $a_{n-2} - a_{n-1}$. So, a recursive rule is given by

$$a_n = a_{n-2} - a_{n-1}.$$

28. ✗ Beginning with the second term in the sequence, each term a_n equals $a_{n-1} - 3$. So, a recursive rule is given by

$$a_1 = 5, a_n = a_{n-1} - 3.$$

In Exercises 29–38, write a recursive rule for the sequence. *(See Example 4.)*

29. $a_n = 3 + 4n$
30. $a_n = -2 - 8n$
31. $a_n = 12 - 10n$
32. $a_n = 9 - 5n$
33. $a_n = 12(11)^{n-1}$
34. $a_n = -7(6)^{n-1}$
35. $a_n = 2.5 - 0.6n$
36. $a_n = -1.4 + 0.5n$
37. $a_n = -\frac{1}{2}\left(\frac{1}{4}\right)^{n-1}$
38. $a_n = \frac{1}{4}(5)^{n-1}$

39. **REWRITING A FORMULA** You have saved \$82 to buy a bicycle. You save an additional \$30 each month. The explicit rule $a_n = 30n + 82$ gives the amount saved after n months. Write a recursive rule for the amount you have saved n months from now.

40. **REWRITING A FORMULA** Your salary is given by the explicit rule $a_n = 35{,}000(1.04)^{n-1}$, where n is the number of years you have worked. Write a recursive rule for your salary.

In Exercises 41–48, write an explicit rule for the sequence. *(See Example 5.)*

41. $a_1 = 3, a_n = a_{n-1} - 6$

42. $a_1 = 16, a_n = a_{n-1} + 7$

43. $a_1 = -2, a_n = 3a_{n-1}$

44. $a_1 = 13, a_n = 4a_{n-1}$

45. $a_1 = -12, a_n = a_{n-1} + 9.1$

46. $a_1 = -4, a_n = 0.65a_{n-1}$

47. $a_1 = 5, a_n = a_{n-1} - \frac{1}{3}$

48. $a_1 = -5, a_n = \frac{1}{4}a_{n-1}$

49. **REWRITING A FORMULA** A grocery store arranges cans in a pyramid-shaped display with 20 cans in the bottom row and two fewer cans in each subsequent row going up. The number of cans in each row is represented by the recursive rule $a_1 = 20$, $a_n = a_{n-1} - 2$. Write an explicit rule for the number of cans in row n.

50. **REWRITING A FORMULA** The value of a car is given by the recursive rule $a_1 = 25{,}600$, $a_n = 0.86a_{n-1}$, where n is the number of years since the car was new. Write an explicit rule for the value of the car after n years.

51. **USING STRUCTURE** What is the 1000th term of the sequence whose first term is $a_1 = 4$ and whose nth term is $a_n = a_{n-1} + 6$? Justify your answer.

(A) 4006 (B) 5998

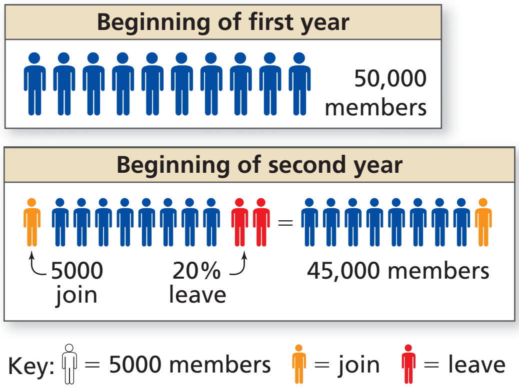
(C) 1010 (D) 10,000

52. **USING STRUCTURE** What is the 873rd term of the sequence whose first term is $a_1 = 0.01$ and whose nth term is $a_n = 1.01a_{n-1}$? Justify your answer.

(A) 58.65 (B) 8.73

(C) 1.08 (D) 586,459.38

53. **PROBLEM SOLVING** An online music service initially has 50,000 members. Each year, the company loses 20% of its current members and gains 5000 new members. *(See Example 6.)*

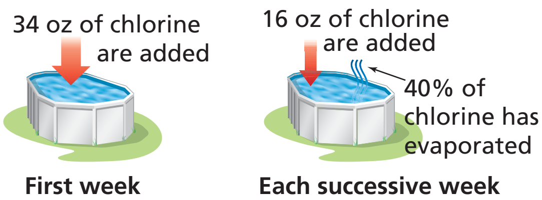


a. Write a recursive rule for the number a_n of members at the start of the nth year.

b. Find the number of members at the start of the fifth year.

c. Describe what happens to the number of members over time.

54. **PROBLEM SOLVING** You add chlorine to a swimming pool. You add 34 ounces of chlorine the first week and 16 ounces every week thereafter. Each week, 40% of the chlorine in the pool evaporates.



a. Write a recursive rule for the amount of chlorine in the pool at the start of the nth week.

b. Find the amount of chlorine in the pool at the start of the third week.

c. Describe what happens to the amount of chlorine in the pool over time.

55. **OPEN-ENDED** Give an example of a real-life situation which you can represent with a recursive rule that does not approach a limit. Write a recursive rule that represents the situation.

56. **OPEN-ENDED** Give an example of a sequence in which each term after the third term is a function of the three terms preceding it. Write a recursive rule for the sequence and find its first eight terms.

57. **MODELING WITH MATHEMATICS** You borrow \$2000 at 9% annual interest compounded monthly for 2 years. The monthly payment is \$91.37. *(See Example 7.)*

a. Find the balance after the fifth payment.

b. Find the amount of the last payment.

58. **MODELING WITH MATHEMATICS** You borrow \$10,000 to build an extra bedroom onto your house. The loan is secured for 7 years at an annual interest rate of 11.5%. The monthly payment is \$173.86.

a. Find the balance after the fourth payment.

b. Find the amount of the last payment.

59. **COMPARING METHODS** In 1202, the mathematician Leonardo Fibonacci wrote *Liber Abaci,* in which he proposed the following rabbit problem:

Begin with a pair of newborn rabbits. When a pair of rabbits is two months old, the rabbits begin producing a new pair of rabbits each month. Assume none of the rabbits die.

Month	1	2	3	4	5	6
Pairs at start of month	1	1	2	3	5	8

This problem produces a sequence called the Fibonacci sequence, which has both a recursive formula and an explicit formula as follows.

Recursive: $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}$

Explicit: $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n,\ n \geq 1$

Use each formula to determine how many rabbits there will be after one year. Justify your answers.

60. **USING TOOLS** A town library initially has 54,000 books in its collection. Each year, 2% of the books are lost or discarded. The library can afford to purchase 1150 new books each year.

a. Write a recursive rule for the number a_n of books in the library at the beginning of the nth year.

b. Use the *sequence* mode and the *dot* mode of a graphing calculator to graph the sequence. What happens to the number of books in the library over time? Explain.

61. **DRAWING CONCLUSIONS** A tree farm initially has 9000 trees. Each year, 10% of the trees are harvested and 800 seedlings are planted.

a. Write a recursive rule for the number of trees on the tree farm at the beginning of the nth year.

b. What happens to the number of trees after an extended period of time?

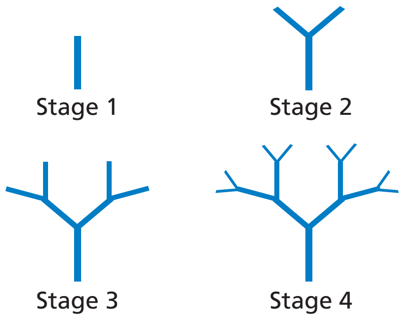
62. **DRAWING CONCLUSIONS** You sprain your ankle and your doctor prescribes 325 milligrams of an anti-inflammatory drug every 8 hours for 10 days. Sixty percent of the drug is removed from the bloodstream every 8 hours.

a. Write a recursive rule for the amount of the drug in the bloodstream after n doses.

b. The value that a drug level approaches after an extended period of time is called the *maintenance level*. What is the maintenance level of this drug given the prescribed dosage?

c. How does doubling the dosage affect the maintenance level of the drug? Justify your answer.

63. **FINDING A PATTERN** A fractal tree starts with a single branch (the trunk). At each stage, each new branch from the previous stage grows two more branches, as shown.



a. List the number of new branches in each of the first seven stages. What type of sequence do these numbers form?

b. Write an explicit rule and a recursive rule for the sequence in part (a).

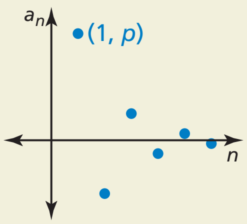
64. **THOUGHT PROVOKING** Let $a_1 = 34$. Then write the terms of the sequence until you discover a pattern.

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n, & \text{if } a_n \text{ is even} \\ 3a_n + 1, & \text{if } a_n \text{ is odd} \end{cases}$$

Do the same for $a_1 = 25$. What can you conclude?

65. **MODELING WITH MATHEMATICS** You make a \$500 down payment on a \$3500 diamond ring. You borrow the remaining balance at 10% annual interest compounded monthly. The monthly payment is \$213.59. How long does it take to pay back the loan? What is the amount of the last payment? Justify your answers.

66. **HOW DO YOU SEE IT?** The graph shows the first six terms of the sequence $a_1 = p$, $a_n = ra_{n-1}$.



a. Describe what happens to the values in the sequence as n increases.

b. Describe the set of possible values for r. Explain your reasoning.

67. **REASONING** The rule for a recursive sequence is as follows.

$$f(1) = 3, f(2) = 10$$
$$f(n) = 4 + 2f(n - 1) - f(n - 2)$$

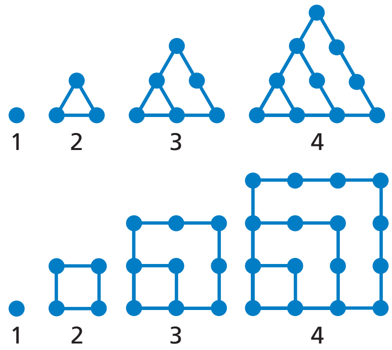
a. Write the first five terms of the sequence.

b. Use finite differences to find a pattern. What type of relationship do the terms of the sequence show?

c. Write an explicit rule for the sequence.

68. **MAKING AN ARGUMENT** Your friend says it is impossible to write a recursive rule for a sequence that is neither arithmetic nor geometric. Is your friend correct? Justify your answer.

69. **CRITICAL THINKING** The first four triangular numbers T_n and the first four square numbers S_n are represented by the points in each diagram.



a. Write an explicit rule for each sequence.

b. Write a recursive rule for each sequence.

c. Write a rule for the square numbers in terms of the triangular numbers. Draw diagrams to explain why this rule is true.

70. **CRITICAL THINKING** You are saving money for retirement. You plan to withdraw \$30,000 at the beginning of each year for 20 years after you retire. Based on the type of investment you are making, you can expect to earn an annual return of 8% on your savings after you retire.

a. Let a_n be your balance n years after retiring. Write a recursive equation that shows how a_n is related to a_{n-1}.

b. Solve the equation from part (a) for a_{n-1}. Find a_0, the minimum amount of money you should have in your account when you retire. (*Hint:* Let $a_{20} = 0$.)

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 5.4)*

71. $\sqrt{x} + 2 = 7$

72. $2\sqrt{x} - 5 = 15$

73. $\sqrt[3]{x} + 16 = 19$

74. $2\sqrt[3]{x} - 13 = -5$

The variables x and y vary inversely. Use the given values to write an equation relating x and y. Then find y when $x = 4$. *(Section 7.1)*

75. $x = 2, y = 9$

76. $x = -4, y = 3$

77. $x = 10, y = 32$