2.2 Characteristics of Quadratic Functions

COMMON CORE

Learning Standards
HSF-IF.B.4
HSF-IF.C.7c
HSF-IF.C.9
HSA-APR.B.3

Essential Question

What type of symmetry does the graph of $f(x) = a(x - h)^2 + k$ have and how can you describe this symmetry?

EXPLORATION 1 Parabolas and Symmetry

Work with a partner.

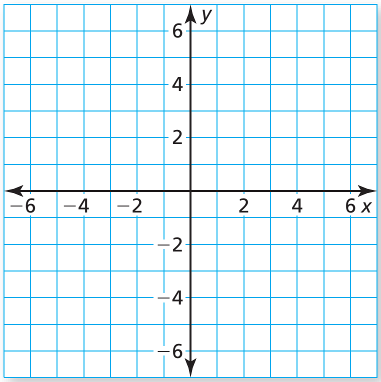
a. Complete the table. Then use the values in the table to sketch the graph of the function

$$f(x) = \frac{1}{2}x^2 - 2x - 2$$

on graph paper.

x	-2	-1	0	1	2
$f(x)$					

x	3	4	5	6
$f(x)$				



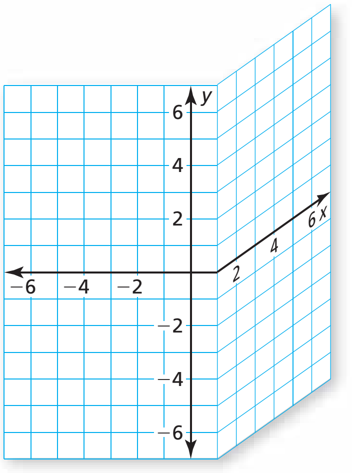
b. Use the results in part (a) to identify the vertex of the parabola.

c. Find a vertical line on your graph paper so that when you fold the paper, the left portion of the graph coincides with the right portion of the graph. What is the equation of this line? How does it relate to the vertex?

d. Show that the vertex form

$$f(x) = \frac{1}{2}(x - 2)^2 - 4$$

is equivalent to the function given in part (a).



EXPLORATION 2 Parabolas and Symmetry

Work with a partner. Repeat Exploration 1 for the function given by

$$f(x) = -\frac{1}{3}x^2 + 2x + 3 = -\frac{1}{3}(x - 3)^2 + 6.$$

ATTENDING TO PRECISION

To be proficient in math, you need to use clear definitions in your reasoning and discussions with others.

Communicate Your Answer

3. What type of symmetry does the graph of the parabola $f(x) = a(x - h)^2 + k$ have and how can you describe this symmetry?

4. Describe the symmetry of each graph. Then use a graphing calculator to verify your answer.

a. $f(x) = -(x - 1)^2 + 4$ **b.** $f(x) = (x + 1)^2 - 2$ **c.** $f(x) = 2(x - 3)^2 + 1$

d. $f(x) = \frac{1}{2}(x + 2)^2$ **e.** $f(x) = -2x^2 + 3$ **f.** $f(x) = 3(x - 5)^2 + 2$

2.2 Lesson

Core Vocabulary

axis of symmetry, *p. 56*
standard form, *p. 56*
minimum value, *p. 58*
maximum value, *p. 58*
intercept form, *p. 59*

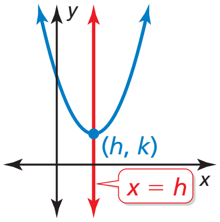
Previous
x-intercept

What You Will Learn

- Explore properties of parabolas.
- Find maximum and minimum values of quadratic functions.
- Graph quadratic functions using *x*-intercepts.
- Solve real-life problems.

Exploring Properties of Parabolas

An **axis of symmetry** is a line that divides a parabola into mirror images and passes through the vertex. Because the vertex of $f(x) = a(x - h)^2 + k$ is (h, k), the axis of symmetry is the vertical line $x = h$.



Previously, you used transformations to graph quadratic functions in vertex form. You can also use the axis of symmetry and the vertex to graph quadratic functions written in vertex form.

EXAMPLE 1 Using Symmetry to Graph Quadratic Functions

Graph $f(x) = -2(x + 3)^2 + 4$. Label the vertex and axis of symmetry.

SOLUTION

Step 1 Identify the constants $a = -2$, $h = -3$, and $k = 4$.

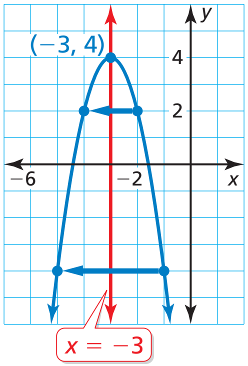
Step 2 Plot the vertex $(h, k) = (-3, 4)$ and draw the axis of symmetry $x = -3$.

Step 3 Evaluate the function for two values of x.

$$x = -2: \; f(-2) = -2(-2 + 3)^2 + 4 = 2$$
$$x = -1: \; f(-1) = -2(-1 + 3)^2 + 4 = -4$$

Plot the points $(-2, 2)$, $(-1, -4)$, and their reflections in the axis of symmetry.

Step 4 Draw a parabola through the plotted points.



Quadratic functions can also be written in **standard form**, $f(x) = ax^2 + bx + c$, where $a \neq 0$. You can derive standard form by expanding vertex form.

$f(x) = a(x - h)^2 + k$ Vertex form

$f(x) = a(x^2 - 2hx + h^2) + k$ Expand $(x - h)^2$.

$f(x) = ax^2 - 2ahx + ah^2 + k$ Distributive Property

$f(x) = ax^2 + (-2ah)x + (ah^2 + k)$ Group like terms.

$f(x) = ax^2 + bx + c$ Let $b = -2ah$ and let $c = ah^2 + k$.

This allows you to make the following observations.

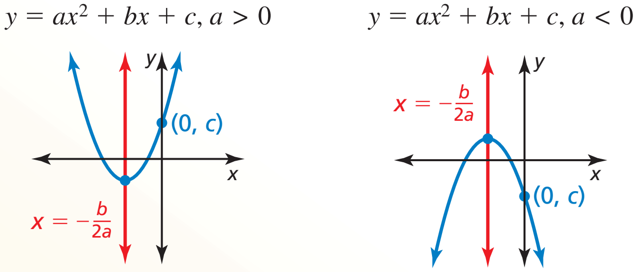
$a = a$: So, a has the same meaning in vertex form and standard form.

$b = -2ah$: Solve for h to obtain $h = -\frac{b}{2a}$. So, the axis of symmetry is $x = -\frac{b}{2a}$.

$c = ah^2 + k$: In vertex form $f(x) = a(x - h)^2 + k$, notice that $f(0) = ah^2 + k$. So, c is the y-intercept.

Core Concept

Properties of the Graph of $f(x) = ax^2 + bx + c$



- The parabola opens up when $a > 0$ and opens down when $a < 0$.
- The graph is narrower than the graph of $f(x) = x^2$ when $|a| > 1$ and wider when $|a| < 1$.
- The axis of symmetry is $x = -\frac{b}{2a}$ and the vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- The y-intercept is c. So, the point $(0, c)$ is on the parabola.

EXAMPLE 2 Graphing a Quadratic Function in Standard Form

Graph $f(x) = 3x^2 - 6x + 1$. Label the vertex and axis of symmetry.

COMMON ERROR

Be sure to include the negative sign when writing the expression for the *x*-coordinate of the vertex.

SOLUTION

Step 1 Identify the coefficients $a = 3$, $b = -6$, and $c = 1$. Because $a > 0$, the parabola opens up.

Step 2 Find the vertex. First calculate the x-coordinate.

$$x = -\frac{b}{2a} = -\frac{-6}{2(3)} = 1$$

Then find the y-coordinate of the vertex.

$$f(1) = 3(1)^2 - 6(1) + 1 = -2$$

So, the vertex is $(1, -2)$. Plot this point.

Step 3 Draw the axis of symmetry $x = 1$.

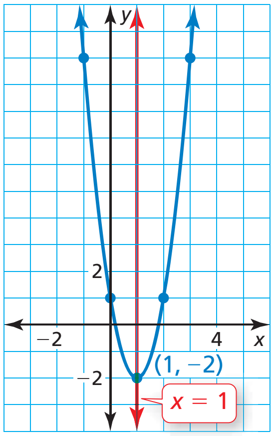
Step 4 Identify the y-intercept c, which is 1. Plot the point $(0, 1)$ and its reflection in the axis of symmetry, $(2, 1)$.

Step 5 Evaluate the function for another value of x, such as $x = 3$.

$$f(3) = 3(3)^2 - 6(3) + 1 = 10$$

Plot the point $(3, 10)$ and its reflection in the axis of symmetry, $(-1, 10)$.

Step 6 Draw a parabola through the plotted points.



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Graph the function. Label the vertex and axis of symmetry.

1. $f(x) = -3(x + 1)^2$

2. $g(x) = 2(x - 2)^2 + 5$

3. $h(x) = x^2 + 2x - 1$

4. $p(x) = -2x^2 - 8x + 1$

Finding Maximum and Minimum Values

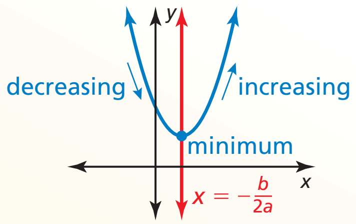
Because the vertex is the highest or lowest point on a parabola, its y-coordinate is the *maximum value* or *minimum value* of the function. The vertex lies on the axis of symmetry, so the function is *increasing* on one side of the axis of symmetry and *decreasing* on the other side.

Core Concept

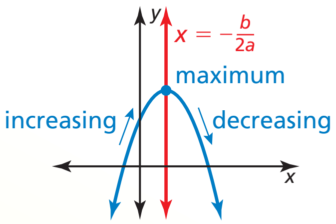
Minimum and Maximum Values

For the quadratic function $f(x) = ax^2 + bx + c$, the y-coordinate of the vertex is the **minimum value** of the function when $a > 0$ and the **maximum value** when $a < 0$.

$a > 0$



$a < 0$



- Minimum value: $f\left(-\frac{b}{2a}\right)$
- Domain: All real numbers
- Range: $y \geq f\left(-\frac{b}{2a}\right)$
- Decreasing to the left of $x = -\frac{b}{2a}$
- Increasing to the right of $x = -\frac{b}{2a}$

- Maximum value: $f\left(-\frac{b}{2a}\right)$
- Domain: All real numbers
- Range: $y \leq f\left(-\frac{b}{2a}\right)$
- Increasing to the left of $x = -\frac{b}{2a}$
- Decreasing to the right of $x = -\frac{b}{2a}$

STUDY TIP

When a function f is written in vertex form, you can use $h = -\frac{b}{2a}$ and $k = f\left(-\frac{b}{2a}\right)$ to state the properties shown.

EXAMPLE 3 Finding a Minimum or a Maximum Value

Find the minimum value or maximum value of $f(x) = \frac{1}{2}x^2 - 2x - 1$. Describe the domain and range of the function, and where the function is increasing and decreasing.

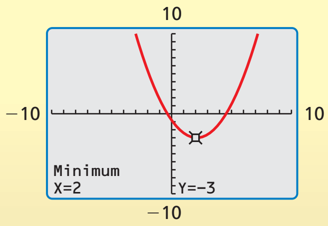
SOLUTION

Identify the coefficients $a = \frac{1}{2}$, $b = -2$, and $c = -1$. Because $a > 0$, the parabola opens up and the function has a minimum value. To find the minimum value, calculate the coordinates of the vertex.

$$x = -\frac{b}{2a} = -\frac{-2}{2\left(\frac{1}{2}\right)} = 2 \quad ➡ \quad f(2) = \frac{1}{2}(2)^2 - 2(2) - 1 = -3$$

▶ The minimum value is -3. So, the domain is all real numbers and the range is $y \geq -3$. The function is decreasing to the left of $x = 2$ and increasing to the right of $x = 2$.

Check



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5. Find the minimum value or maximum value of (a) $f(x) = 4x^2 + 16x - 3$ and (b) $h(x) = -x^2 + 5x + 9$. Describe the domain and range of each function, and where each function is increasing and decreasing.

Graphing Quadratic Functions Using x-Intercepts

When the graph of a quadratic function has at least one x-intercept, the function can be written in **intercept form**, $f(x) = a(x - p)(x - q)$, where $a \neq 0$.

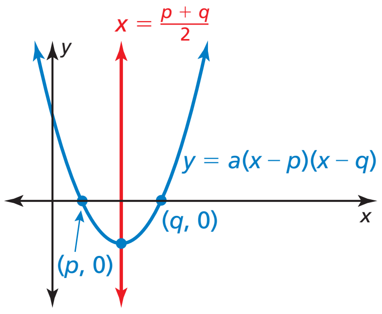
REMEMBER

An x-intercept of a graph is the x-coordinate of a point where the graph intersects the x-axis. It occurs where $f(x) = 0$.

Core Concept

Properties of the Graph of $f(x) = a(x - p)(x - q)$

- Because $f(p) = 0$ and $f(q) = 0$, p and q are the x-intercepts of the graph of the function.
- The axis of symmetry is halfway between $(p, 0)$ and $(q, 0)$. So, the axis of symmetry is $x = \dfrac{p + q}{2}$.
- The parabola opens up when $a > 0$ and opens down when $a < 0$.



COMMON ERROR

Remember that the x-intercepts of the graph of $f(x) = a(x - p)(x - q)$ are p and q, not $-p$ and $-q$.

EXAMPLE 4 Graphing a Quadratic Function in Intercept Form

Graph $f(x) = -2(x + 3)(x - 1)$. Label the x-intercepts, vertex, and axis of symmetry.

SOLUTION

Step 1 Identify the x-intercepts. The x-intercepts are $p = -3$ and $q = 1$, so the parabola passes through the points $(-3, 0)$ and $(1, 0)$.

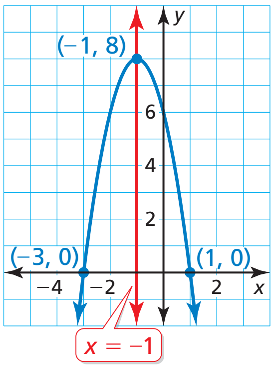
Step 2 Find the coordinates of the vertex.

$$x = \frac{p + q}{2} = \frac{-3 + 1}{2} = -1$$

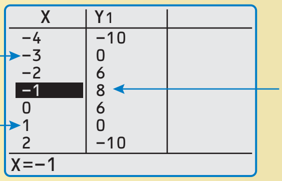
$$f(-1) = -2(-1 + 3)(-1 - 1) = 8$$

So, the axis of symmetry is $x = -1$ and the vertex is $(-1, 8)$.

Step 3 Draw a parabola through the vertex and the points where the x-intercepts occur.



Check You can check your answer by generating a table of values for f on a graphing calculator.



X	Y1
-4	-10
-3	0
-2	6
-1	8
0	6
1	0
2	-10

X=-1

x-intercept

x-intercept

The values show symmetry about $x = -1$. So, the vertex is $(-1, 8)$.

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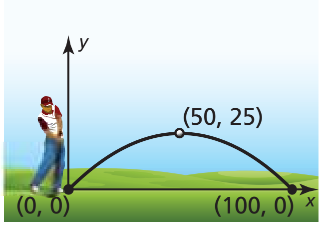
Graph the function. Label the x-intercepts, vertex, and axis of symmetry.

6. $f(x) = -(x + 1)(x + 5)$

7. $g(x) = \frac{1}{4}(x - 6)(x - 2)$

Solving Real-Life Problems

EXAMPLE 5 Modeling with Mathematics

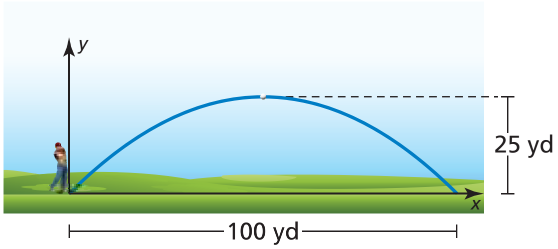


The parabola shows the path of your first golf shot, where x is the horizontal distance (in yards) and y is the corresponding height (in yards). The path of your second shot can be modeled by the function $f(x) = -0.02x(x - 80)$. Which shot travels farther before hitting the ground? Which travels higher?

SOLUTION

1. **Understand the Problem** You are given a graph and a function that represent the paths of two golf shots. You are asked to determine which shot travels farther before hitting the ground and which shot travels higher.
2. **Make a Plan** Determine how far each shot travels by interpreting the x-intercepts. Determine how high each shot travels by finding the maximum value of each function. Then compare the values.
3. **Solve the Problem**

First shot: The graph shows that the x-intercepts are 0 and 100. So, the ball travels 100 yards before hitting the ground.



Because the axis of symmetry is halfway between (0, 0) and (100, 0), the axis of symmetry is $x = \frac{0 + 100}{2} = 50$. So, the vertex is (50, 25) and the maximum height is 25 yards.

Second shot: By rewriting the function in intercept form as $f(x) = -0.02(x - 0)(x - 80)$, you can see that $p = 0$ and $q = 80$. So, the ball travels 80 yards before hitting the ground.

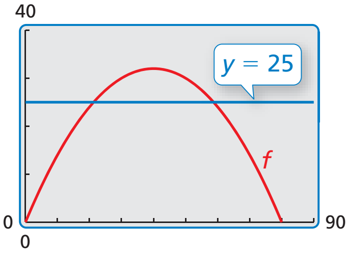
To find the maximum height, find the coordinates of the vertex.

$$x = \frac{p + q}{2} = \frac{0 + 80}{2} = 40$$

$$f(40) = -0.02(40)(40 - 80) = 32$$

The maximum height of the second shot is 32 yards.

▶ Because 100 yards > 80 yards, the first shot travels farther. Because 32 yards > 25 yards, the second shot travels higher.



4. **Look Back** To check that the second shot travels higher, graph the function representing the path of the second shot and the line $y = 25$, which represents the maximum height of the first shot.

The graph rises above $y = 25$, so the second shot travels higher. ✓

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8. **WHAT IF?** The graph of your third shot is a parabola through the origin that reaches a maximum height of 28 yards when $x = 45$. Compare the distance it travels before it hits the ground with the distances of the first two shots.

2.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain how to determine whether a quadratic function will have a minimum value or a maximum value.

2. **WHICH ONE DOESN'T BELONG?** The graph of which function does *not* belong with the other three? Explain.

$f(x) = 3x^2 + 6x - 24$

$f(x) = 3x^2 + 24x - 6$

$f(x) = 3(x - 2)(x + 4)$

$f(x) = 3(x + 1)^2 - 27$

Monitoring Progress and Modeling with Mathematics

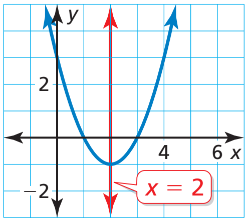
In Exercises 3–14, graph the function. Label the vertex and axis of symmetry. *(See Example 1.)*

3. $f(x) = (x - 3)^2$
4. $h(x) = (x + 4)^2$
5. $g(x) = (x + 3)^2 + 5$
6. $y = (x - 7)^2 - 1$
7. $y = -4(x - 2)^2 + 4$
8. $g(x) = 2(x + 1)^2 - 3$
9. $f(x) = -2(x - 1)^2 - 5$
10. $h(x) = 4(x + 4)^2 + 6$
11. $y = -\frac{1}{4}(x + 2)^2 + 1$
12. $y = \frac{1}{2}(x - 3)^2 + 2$
13. $f(x) = 0.4(x - 1)^2$
14. $g(x) = 0.75x^2 - 5$

ANALYZING RELATIONSHIPS In Exercises 15–18, use the axis of symmetry to match the equation with its graph.

15. $y = 2(x - 3)^2 + 1$
16. $y = (x + 4)^2 - 2$
17. $y = \frac{1}{2}(x + 1)^2 + 3$
18. $y = (x - 2)^2 - 1$

A.



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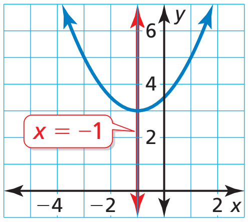

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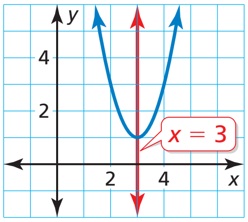

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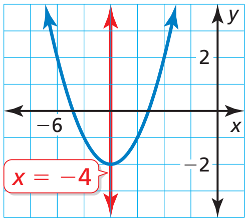
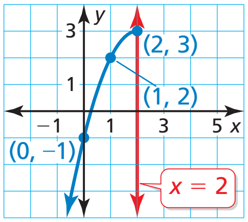
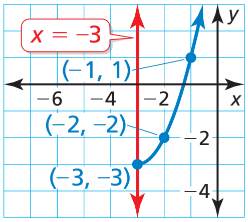


REASONING In Exercises 19 and 20, use the axis of symmetry to plot the reflection of each point and complete the parabola.

19.



20.



In Exercises 21–30, graph the function. Label the vertex and axis of symmetry. *(See Example 2.)*

21. $y = x^2 + 2x + 1$
22. $y = 3x^2 - 6x + 4$
23. $y = -4x^2 + 8x + 2$
24. $f(x) = -x^2 - 6x + 3$
25. $g(x) = -x^2 - 1$
26. $f(x) = 6x^2 - 5$
27. $g(x) = -1.5x^2 + 3x + 2$
28. $f(x) = 0.5x^2 + x - 3$
29. $y = \frac{3}{2}x^2 - 3x + 6$
30. $y = -\frac{5}{2}x^2 - 4x - 1$

31. **WRITING** Two quadratic functions have graphs with vertices (2, 4) and (2, −3). Explain why you can not use the axes of symmetry to distinguish between the two functions.

32. **WRITING** A quadratic function is increasing to the left of $x = 2$ and decreasing to the right of $x = 2$. Will the vertex be the highest or lowest point on the graph of the parabola? Explain.

ERROR ANALYSIS **In Exercises 33 and 34, describe and correct the error in analyzing the graph of $y = 4x^2 + 24x - 7$.**

33. ✗ The x-coordinate of the vertex is

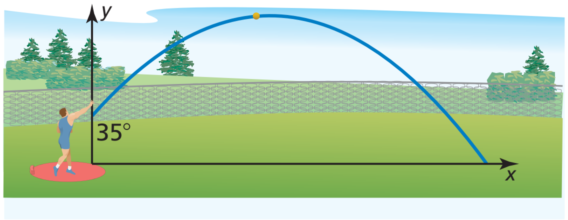
$$x = \frac{b}{2a} = \frac{24}{2(4)} = 3.$$

34. ✗ The y-intercept of the graph is the value of c, which is 7.

MODELING WITH MATHEMATICS **In Exercises 35 and 36, x is the horizontal distance (in feet) and y is the vertical distance (in feet). Find and interpret the coordinates of the vertex.**

35. The path of a basketball thrown at an angle of 45° can be modeled by $y = -0.02x^2 + x + 6$.

36. The path of a shot put released at an angle of 35° can be modeled by $y = -0.01x^2 + 0.7x + 6$.



37. **ANALYZING EQUATIONS** The graph of which function has the same axis of symmetry as the graph of $y = x^2 + 2x + 2$?

Ⓐ $y = 2x^2 + 2x + 2$

Ⓑ $y = -3x^2 - 6x + 2$

Ⓒ $y = x^2 - 2x + 2$

Ⓓ $y = -5x^2 + 10x + 2$

38. **USING STRUCTURE** Which function represents the parabola with the widest graph? Explain your reasoning.

Ⓐ $y = 2(x + 3)^2$

Ⓑ $y = x^2 - 5$

Ⓒ $y = 0.5(x - 1)^2 + 1$

Ⓓ $y = -x^2 + 6$

In Exercises 39–48, find the minimum or maximum value of the function. Describe the domain and range of the function, and where the function is increasing and decreasing. *(See Example 3.)*

39. $y = 6x^2 - 1$

40. $y = 9x^2 + 7$

41. $y = -x^2 - 4x - 2$

42. $g(x) = -3x^2 - 6x + 5$

43. $f(x) = -2x^2 + 8x + 7$

44. $g(x) = 3x^2 + 18x - 5$

45. $h(x) = 2x^2 - 12x$

46. $h(x) = x^2 - 4x$

47. $y = \frac{1}{4}x^2 - 3x + 2$

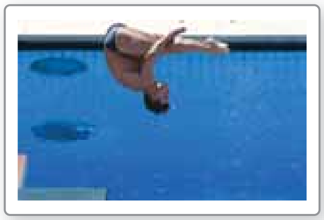
48. $f(x) = \frac{3}{2}x^2 + 6x + 4$

49. **PROBLEM SOLVING** The path of a diver is modeled by the function $f(x) = -9x^2 + 9x + 1$, where $f(x)$ is the height of the diver (in meters) above the water and x is the horizontal distance (in meters) from the end of the diving board.

a. What is the height of the diving board?

b. What is the maximum height of the diver?

c. Describe where the diver is ascending and where the diver is descending.



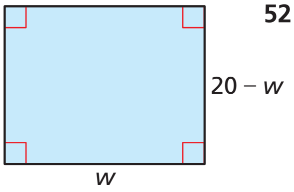
50. **PROBLEM SOLVING** The engine torque y (in foot-pounds) of one model of car is given by $y = -3.75x^2 + 23.2x + 38.8$, where x is the speed (in thousands of revolutions per minute) of the engine.

a. Find the engine speed that maximizes torque. What is the maximum torque?

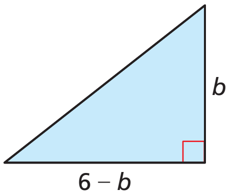
b. Explain what happens to the engine torque as the speed of the engine increases.

MATHEMATICAL CONNECTIONS **In Exercises 51 and 52, write an equation for the area of the figure. Then determine the maximum possible area of the figure.**

51.



52.



In Exercises 53–60, graph the function. Label the x-intercept(s), vertex, and axis of symmetry. *(See Example 4.)*

53. $y = (x + 3)(x - 3)$

54. $y = (x + 1)(x - 3)$

55. $y = 3(x + 2)(x + 6)$

56. $f(x) = 2(x - 5)(x - 1)$

57. $g(x) = -x(x + 6)$

58. $y = -4x(x + 7)$

59. $f(x) = -2(x - 3)^2$

60. $y = 4(x - 7)^2$

USING TOOLS **In Exercises 61–64, identify the x-intercepts of the function and describe where the graph is increasing and decreasing. Use a graphing calculator to verify your answer.**

61. $f(x) = \frac{1}{2}(x - 2)(x + 6)$

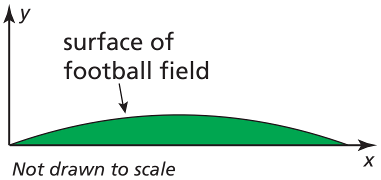
62. $y = \frac{3}{4}(x + 1)(x - 3)$

63. $g(x) = -4(x - 4)(x - 2)$

64. $h(x) = -5(x + 5)(x + 1)$

65. MODELING WITH MATHEMATICS A soccer player kicks a ball downfield. The height of the ball increases until it reaches a maximum height of 8 yards, 20 yards away from the player. A second kick is modeled by $y = x(0.4 - 0.008x)$. Which kick travels farther before hitting the ground? Which kick travels higher? *(See Example 5.)*

66. MODELING WITH MATHEMATICS Although a football field appears to be flat, some are actually shaped like a parabola so that rain runs off to both sides. The cross section of a field can be modeled by $y = -0.000234x(x - 160)$, where x and y are measured in feet. What is the width of the field? What is the maximum height of the surface of the field?



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67. REASONING The points (2, 3) and (−4, 2) lie on the graph of a quadratic function. Determine whether you can use these points to find the axis of symmetry. If not, explain. If so, write the equation of the axis of symmetry.

68. OPEN-ENDED Write two different quadratic functions in intercept form whose graphs have the axis of symmetry $x = 3$.

69. PROBLEM SOLVING An online music store sells about 4000 songs each day when it charges \$1 per song. For each \$0.05 increase in price, about 80 fewer songs per day are sold. Use the verbal model and quadratic function to determine how much the store should charge per song to maximize daily revenue.

Revenue (dollars) = Price (dollars/song) • Sales (songs)

$$R(x) = (1 + 0.05x) \cdot (4000 - 80x)$$

70. PROBLEM SOLVING An electronics store sells 70 digital cameras per month at a price of \$320 each. For each \$20 decrease in price, about 5 more cameras per month are sold. Use the verbal model and quadratic function to determine how much the store should charge per camera to maximize monthly revenue.

Revenue (dollars) = Price (dollars/camera) • Sales (cameras)

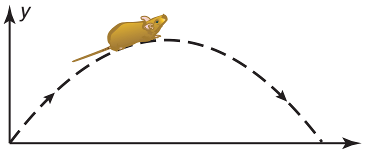
$$R(x) = (320 - 20x) \cdot (70 + 5x)$$

71. DRAWING CONCLUSIONS Compare the graphs of the three quadratic functions. What do you notice? Rewrite the functions f and g in standard form to justify your answer.

$$f(x) = (x + 3)(x + 1)$$
$$g(x) = (x + 2)^2 - 1$$
$$h(x) = x^2 + 4x + 3$$

72. USING STRUCTURE Write the quadratic function $f(x) = x^2 + x - 12$ in intercept form. Graph the function. Label the x-intercepts, y-intercept, vertex, and axis of symmetry.

73. PROBLEM SOLVING A woodland jumping mouse hops along a parabolic path given by $y = -0.2x^2 + 1.3x$, where x is the mouse's horizontal distance traveled (in feet) and y is the corresponding height (in feet). Can the mouse jump over a fence that is 3 feet high? Justify your answer.



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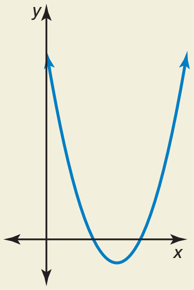
74. **HOW DO YOU SEE IT?** Consider the graph of the function $f(x) = a(x - p)(x - q)$.



a. What does $f\left(\frac{p+q}{2}\right)$ represent in the graph?

b. If $a < 0$, how does your answer in part (a) change? Explain.

75. **MODELING WITH MATHEMATICS** The Gateshead Millennium Bridge spans the River Tyne. The arch of the bridge can be modeled by a parabola. The arch reaches a maximum height of 50 meters at a point roughly 63 meters across the river. Graph the curve of the arch. What are the domain and range? What do they represent in this situation?

76. **THOUGHT PROVOKING** You have 100 feet of fencing to enclose a rectangular garden. Draw three possible designs for the garden. Of these, which has the greatest area? Make a conjecture about the dimensions of the rectangular garden with the greatest possible area. Explain your reasoning.

77. **MAKING AN ARGUMENT** The point (1, 5) lies on the graph of a quadratic function with axis of symmetry $x = -1$. Your friend says the vertex could be the point (0, 5). Is your friend correct? Explain.

78. **CRITICAL THINKING** Find the y-intercept in terms of a, p, and q for the quadratic function $f(x) = a(x - p)(x - q)$.

79. **MODELING WITH MATHEMATICS** A kernel of popcorn contains water that expands when the kernel is heated, causing it to pop. The equations below represent the “popping volume” y (in cubic centimeters per gram) of popcorn with moisture content x (as a percent of the popcorn’s weight).

Hot-air popping: $y = -0.761(x - 5.52)(x - 22.6)$

Hot-oil popping: $y = -0.652(x - 5.35)(x - 21.8)$

a. For hot-air popping, what moisture content maximizes popping volume? What is the maximum volume?

b. For hot-oil popping, what moisture content maximizes popping volume? What is the maximum volume?

c. Use a graphing calculator to graph both functions in the same coordinate plane. What are the domain and range of each function in this situation? Explain.

80. **ABSTRACT REASONING** A function is written in intercept form with $a > 0$. What happens to the vertex of the graph as a increases? as a approaches 0?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check for extraneous solutions. *(Skills Review Handbook)*

81. $3\sqrt{x} - 6 = 0$

82. $2\sqrt{x - 4} - 2 = 2$

83. $\sqrt{5x} + 5 = 0$

84. $\sqrt{3x + 8} = \sqrt{x + 4}$

Solve the proportion. *(Skills Review Handbook)*

85. $\frac{1}{2} = \frac{x}{4}$

86. $\frac{2}{3} = \frac{x}{9}$

87. $\frac{-1}{4} = \frac{3}{x}$

88. $\frac{5}{2} = \frac{-20}{x}$