3.5 Solving Nonlinear Systems

COMMON CORE

Learning Standards
HSA-CED.A.3
HSA-REI.C.7
HSA-REI.D.11

Essential Question

How can you solve a nonlinear system of equations?

EXPLORATION 1 Solving Nonlinear Systems of Equations

Work with a partner. Match each system with its graph. Explain your reasoning. Then solve each system using the graph.

a. $y = x^2$
$y = x + 2$

b. $y = x^2 + x - 2$
$y = x + 2$

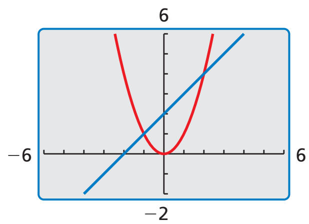
c. $y = x^2 - 2x - 5$
$y = -x + 1$

d. $y = x^2 + x - 6$
$y = -x^2 - x + 6$

e. $y = x^2 - 2x + 1$
$y = -x^2 + 2x - 1$

f. $y = x^2 + 2x + 1$
$y = -x^2 + x + 2$

A.



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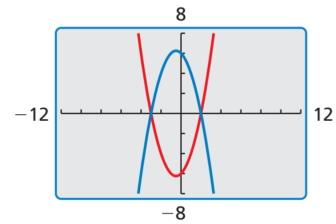

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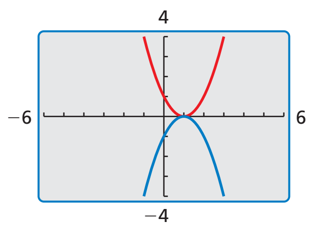

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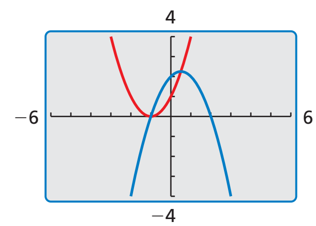

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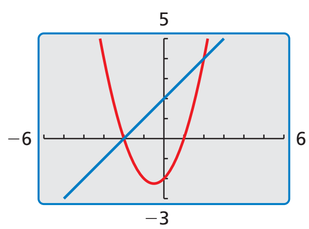
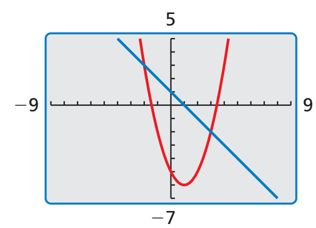


F.



MAKING SENSE OF PROBLEMS

To be proficient in math, you need to plan a solution pathway rather than simply jumping into a solution attempt.

EXPLORATION 2 Solving Nonlinear Systems of Equations

Work with a partner. Look back at the nonlinear system in Exploration 1(f). Suppose you want a more accurate way to solve the system than using a graphical approach.

a. Show how you could use a *numerical approach* by creating a table. For instance, you might use a spreadsheet to solve the system.

b. Show how you could use an *analytical approach*. For instance, you might try solving the system by substitution or elimination.

Communicate Your Answer

3. How can you solve a nonlinear system of equations?

4. Would you prefer to use a graphical, numerical, or analytical approach to solve the given nonlinear system of equations? Explain your reasoning.

$y = x^2 + 2x - 3$
$y = -x^2 - 2x + 4$

3.5 Lesson

Core Vocabulary

system of nonlinear equations, *p. 132*

Previous

system of linear equations

circle

What You Will Learn

- Solve systems of nonlinear equations.
- Solve quadratic equations by graphing.

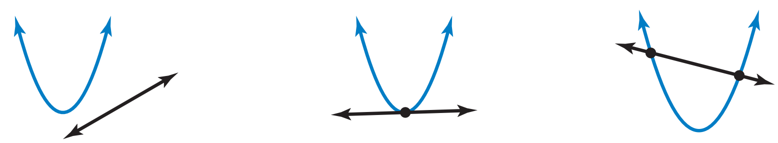
Systems of Nonlinear Equations

Previously, you solved systems of *linear* equations by graphing, substitution, and elimination. You can also use these methods to solve a system of *nonlinear* equations. In a **system of nonlinear equations**, at least one of the equations is nonlinear. For instance, the nonlinear system shown has a quadratic equation and a linear equation.

$y = x^2 + 2x - 4$ Equation 1 is nonlinear.

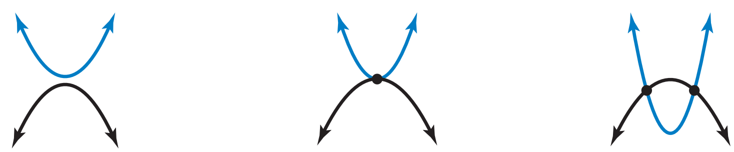
$y = 2x + 5$ Equation 2 is linear.

When the graphs of the equations in a system are a line and a parabola, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions, as shown.



No solution   One solution   Two solutions

When the graphs of the equations in a system are a parabola that opens up and a parabola that opens down, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions, as shown.



No solution   One solution   Two solutions

EXAMPLE 1 Solving a Nonlinear System by Graphing

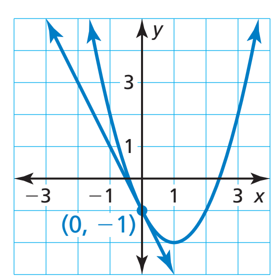
Solve the system by graphing.

$y = x^2 - 2x - 1$ Equation 1

$y = -2x - 1$ Equation 2

SOLUTION

Graph each equation. Then estimate the point of intersection. The parabola and the line appear to intersect at the point $(0, -1)$. Check the point by substituting the coordinates into each of the original equations.



Equation 1

$y = x^2 - 2x - 1$

$-1 \stackrel{?}{=} (0)^2 - 2(0) - 1$

$-1 = -1$ ✓

Equation 2

$y = -2x - 1$

$-1 \stackrel{?}{=} -2(0) - 1$

$-1 = -1$ ✓

▶ The solution is $(0, -1)$.

EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve the system by substitution.

$x^2 + x - y = -1$ Equation 1

$x + y = 4$ Equation 2

SOLUTION

Begin by solving for y in Equation 2.

$y = -x + 4$ Solve for y in Equation 2.

Next, substitute $-x + 4$ for y in Equation 1 and solve for x.

$x^2 + x - y = -1$ Write Equation 1.

$x^2 + x - (-x + 4) = -1$ Substitute $-x + 4$ for y.

$x^2 + 2x - 4 = -1$ Simplify.

$x^2 + 2x - 3 = 0$ Write in standard form.

$(x + 3)(x - 1) = 0$ Factor.

$x + 3 = 0$ or $x - 1 = 0$ Zero-Product Property

$x = -3$ or $x = 1$ Solve for x.

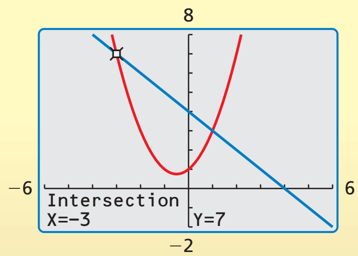
To solve for y, substitute $x = -3$ and $x = 1$ into the equation $y = -x + 4$.

$y = -x + 4 = -(-3) + 4 = 7$ Substitute -3 for x.

$y = -x + 4 = -1 + 4 = 3$ Substitute 1 for x.

▶ The solutions are $(-3, 7)$ and $(1, 3)$. Check the solutions by graphing the system.

Check



EXAMPLE 3 Solving a Nonlinear System by Elimination

Solve the system by elimination.

$2x^2 - 5x - y = -2$ Equation 1

$x^2 + 2x + y = 0$ Equation 2

SOLUTION

Add the equations to eliminate the y-term and obtain a quadratic equation in x.

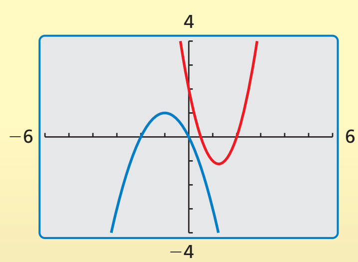
$$\begin{array}{r} 2x^2 - 5x - y = -2 \\ x^2 + 2x + y = 0 \\ \hline 3x^2 - 3x \quad = -2 \end{array}$$ Add the equations.

$3x^2 - 3x + 2 = 0$ Write in standard form.

$x = \dfrac{3 \pm \sqrt{-15}}{6}$ Use the Quadratic Formula.

▶ Because the discriminant is negative, the equation $3x^2 - 3x + 2 = 0$ has no real solution. So, the original system has no real solution. You can check this by graphing the system and seeing that the graphs do not appear to intersect.

Check



Monitoring Progress

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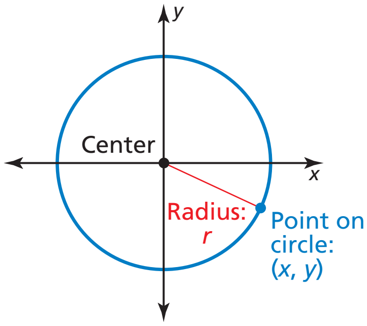
Solve the system using any method. Explain your choice of method.

1. $y = -x^2 + 4$
$y = -4x + 8$

2. $x^2 + 3x + y = 0$
$2x + y = 5$

3. $2x^2 + 4x - y = -2$
$x^2 + y = 2$

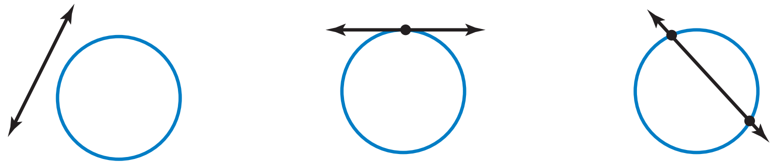


Some nonlinear systems have equations of the form

$$x^2 + y^2 = r^2.$$

This equation is the standard form of a circle with center (0, 0) and radius r.

When the graphs of the equations in a system are a line and a circle, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions, as shown.



EXAMPLE 4 Solving a Nonlinear System by Substitution

Solve the system by substitution.

$x^2 + y^2 = 10$ Equation 1

$y = -3x + 10$ Equation 2

SOLUTION

Substitute $-3x + 10$ for y in Equation 1 and solve for x.

$x^2 + y^2 = 10$ Write Equation 1.

$x^2 + (-3x + 10)^2 = 10$ Substitute $-3x + 10$ for y.

$x^2 + 9x^2 - 60x + 100 = 10$ Expand the power.

$10x^2 - 60x + 90 = 0$ Write in standard form.

$x^2 - 6x + 9 = 0$ Divide each side by 10.

$(x - 3)^2 = 0$ Perfect Square Trinomial Pattern

$x = 3$ Zero-Product Property

COMMON ERROR

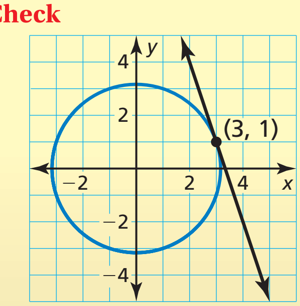
You can also substitute $x = 3$ in Equation 1 to find y. This yields two *apparent* solutions, (3, 1) and (3, −1). However, (3, −1) is *not* a solution because it does not satisfy Equation 2. You can also see (3, −1) is not a solution from the graph.

To find the y-coordinate of the solution, substitute $x = 3$ in Equation 2.

$$y = -3(3) + 10 = 1$$

▶ The solution is (3, 1). Check the solution by graphing the system. You can see that the line and the circle intersect only at the point (3, 1).

Check



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Solve the system.

4. $x^2 + y^2 = 16$
$y = -x + 4$

5. $x^2 + y^2 = 4$
$y = x + 4$

6. $x^2 + y^2 = 1$
$y = \frac{1}{2}x + \frac{1}{2}$

Solving Equations by Graphing

You can solve an equation by rewriting it as a system of equations and then solving the system by graphing.

Core Concept

Solving Equations by Graphing

Step 1 To solve the equation $f(x) = g(x)$, write a system of two equations, $y = f(x)$ and $y = g(x)$.

Step 2 Graph the system of equations $y = f(x)$ and $y = g(x)$. The x-value of each solution of the system is a solution of the equation $f(x) = g(x)$.

ANOTHER WAY

In Example 5(a), you can also find the solutions by writing the given equation as $4x^2 + 3x - 2 = 0$ and solving this equation using the Quadratic Formula.

EXAMPLE 5 Solving Quadratic Equations by Graphing

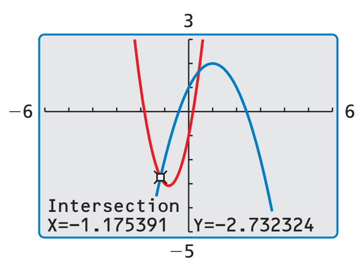
Solve (a) $3x^2 + 5x - 1 = -x^2 + 2x + 1$ and (b) $-(x - 1.5)^2 + 2.25 = 2x(x + 1.5)$ by graphing.

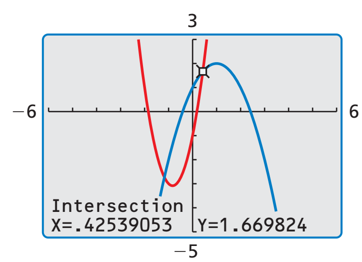
SOLUTION

a. Step 1 Write a system of equations using each side of the original equation.

Equation	*System*
$3x^2 + 5x - 1 = -x^2 + 2x + 1$	$y = 3x^2 + 5x - 1$ $y = -x^2 + 2x + 1$

Step 2 Use a graphing calculator to graph the system. Then use the *intersect* feature to find the x-value of each solution of the system.



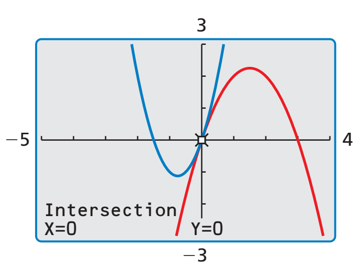


The graphs intersect when $x \approx -1.18$ and $x \approx 0.43$.

▶ The solutions of the equation are $x \approx -1.18$ and $x \approx 0.43$.

b. Step 1 Write a system of equations using each side of the original equation.

Equation	*System*
$-(x - 1.5)^2 + 2.25 = 2x(x + 1.5)$	$y = -(x - 1.5)^2 + 2.25$ $y = 2x(x + 1.5)$



Step 2 Use a graphing calculator to graph the system, as shown at the left. Then use the *intersect* feature to find the x-value of each solution of the system. The graphs intersect when $x = 0$.

▶ The solution of the equation is $x = 0$.

Monitoring Progress

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Solve the equation by graphing.

7. $x^2 - 6x + 15 = -(x - 3)^2 + 6$

8. $(x + 4)(x - 1) = -x^2 + 3x + 4$

3.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe the possible solutions of a system consisting of two quadratic equations.

2. **WHICH ONE DOESN'T BELONG?** Which system does *not* belong with the other three? Explain your reasoning.

$y = 3x + 4$ $y = x^2 + 1$	$y = 2x - 1$ $y = -3x + 6$	$y = 3x^2 + 4x + 1$ $y = -5x^2 - 3x + 1$	$x^2 + y^2 = 4$ $y = -x + 1$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, solve the system by graphing. Check your solution(s). *(See Example 1.)*

3. $y = x + 2$
   $y = 0.5(x + 2)^2$

4. $y = (x - 3)^2 + 5$
   $y = 5$

5. $y = \frac{1}{3}x + 2$
   $y = -3x^2 - 5x - 4$

6. $y = -3x^2 - 30x - 71$
   $y = -3x - 17$

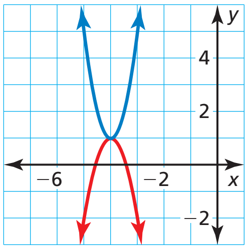
7. $y = x^2 + 8x + 18$
   $y = -2x^2 - 16x - 30$

8. $y = -2x^2 - 9$
   $y = -4x - 1$

9. $y = (x - 2)^2$
   $y = -x^2 + 4x - 2$

10. $y = \frac{1}{2}(x + 2)^2$
    $y = -\frac{1}{2}x^2 + 2$

In Exercises 11–14, solve the system of nonlinear equations using the graph.

11.


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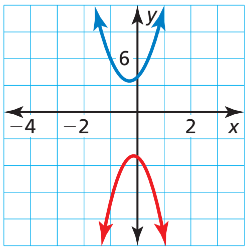

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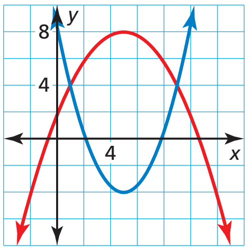

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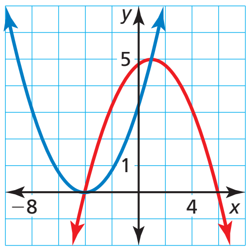


In Exercises 15–24, solve the system by substitution. *(See Examples 2 and 4.)*

15. $y = x + 5$
    $y = x^2 - x + 2$

16. $x^2 + y^2 = 49$
    $y = 7 - x$

17. $x^2 + y^2 = 64$
    $y = -8$

18. $x = 3$
    $-3x^2 + 4x - y = 8$

19. $2x^2 + 4x - y = -3$
    $-2x + y = -4$

20. $2x - 3 = y + 5x^2$
    $y = -3x - 3$

21. $y = x^2 - 1$
    $-7 = -x^2 - y$

22. $y + 16x - 22 = 4x^2$
    $4x^2 - 24x + 26 + y = 0$

23. $x^2 + y^2 = 7$
    $x + 3y = 21$

24. $x^2 + y^2 = 5$
    $-x + y = -1$

25. **USING EQUATIONS** Which ordered pairs are solutions of the nonlinear system?

$$y = \frac{1}{2}x^2 - 5x + \frac{21}{2}$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Ⓐ (1, 6)　　Ⓑ (3, 0)

Ⓒ (8, 2.5)　　Ⓓ (7, 0)

26. **USING EQUATIONS** How many solutions does the system have? Explain your reasoning.

$$y = 7x^2 - 11x + 9$$
$$y = -7x^2 + 5x - 3$$

Ⓐ 0　　Ⓑ 1

Ⓒ 2　　Ⓓ 4

In Exercises 27–34, solve the system by elimination. *(See Example 3.)*

27. $2x^2 - 3x - y = -5$
$-x + y = 5$

28. $-3x^2 + 2x - 5 = y$
$-x + 2 = -y$

29. $-3x^2 + y = -18x + 29$
$-3x^2 - y = 18x - 25$

30. $y = -x^2 - 6x - 10$
$y = 3x^2 + 18x + 22$

31. $y + 2x = -14$
$-x^2 - y - 6x = 11$

32. $y = x^2 + 4x + 7$
$-y = 4x + 7$

33. $y = -3x^2 - 30x - 76$
$y = 2x^2 + 20x + 44$

34. $-10x^2 + y = -80x + 155$
$5x^2 + y = 40x - 85$

35. ERROR ANALYSIS Describe and correct the error in using elimination to solve a system.

$$y = -2x^2 + 32x - 126$$
$$-y = 2x - 14$$
$$0 = 18x - 126$$
$$126 = 18x$$
$$x = 7$$

36. NUMBER SENSE The table shows the inputs and outputs of two quadratic equations. Identify the solution(s) of the system. Explain your reasoning.

x	y_1	y_2
−3	29	−11
−1	9	9
1	−3	21
3	−7	25
7	9	9
11	57	−39

In Exercises 37–42, solve the system using any method. Explain your choice of method.

37. $y = x^2 - 1$
$-y = 2x^2 + 1$

38. $y = -4x^2 - 16x - 13$
$-3x^2 + y + 12x = 17$

39. $-2x + 10 + y = \frac{1}{3}x^2$
$y = 10$

40. $y = 0.5x^2 - 10$
$y = -x^2 + 14$

41. $y = -3(x - 4)^2 + 6$
$(x - 4)^2 + 2 - y = 0$

42. $-x^2 + y^2 = 100$
$y = -x + 14$

USING TOOLS In Exercises 43–48, solve the equation by graphing. *(See Example 5.)*

43. $x^2 + 2x = -\frac{1}{2}x^2 + 2x$

44. $2x^2 - 12x - 16 = -6x^2 + 60x - 144$

45. $(x + 2)(x - 2) = -x^2 + 6x - 7$

46. $-2x^2 - 16x - 25 = 6x^2 + 48x + 95$

47. $(x - 2)^2 - 3 = (x + 3)(-x + 9) - 38$

48. $(-x + 4)(x + 8) - 42 = (x + 3)(x + 1) - 1$

49. REASONING A nonlinear system contains the equations of a constant function and a quadratic function. The system has one solution. Describe the relationship between the graphs.

50. PROBLEM SOLVING The range (in miles) of a broadcast signal from a radio tower is bounded by a circle given by the equation

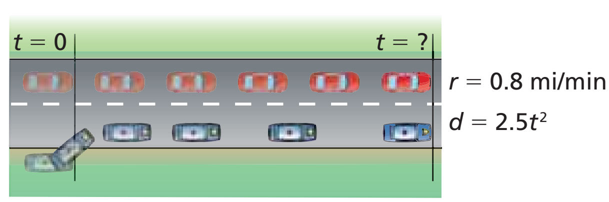
$$x^2 + y^2 = 1620.$$

A straight highway can be modeled by the equation

$$y = -\frac{1}{3}x + 30.$$

For what lengths of the highway are cars able to receive the broadcast signal?

51. PROBLEM SOLVING A car passes a parked police car and continues at a constant speed r. The police car begins accelerating at a constant rate when it is passed. The diagram indicates the distance d (in miles) the police car travels as a function of time t (in minutes) after being passed. Write and solve a system of equations to find how long it takes the police car to catch up to the other car.



52. THOUGHT PROVOKING Write a nonlinear system that has two different solutions with the same y-coordinate. Sketch a graph of your system. Then solve the system.

53. OPEN-ENDED Find three values for m so the system has no solution, one solution, and two solutions. Justify your answer using a graph.

$$3y = -x^2 + 8x - 7$$
$$y = mx + 3$$

54. MAKING AN ARGUMENT You and a friend solve the system shown and determine that $x = 3$ and $x = -3$. You use Equation 1 to obtain the solutions $(3, 3)$, $(3, -3)$, $(-3, 3)$, and $(-3, -3)$. Your friend uses Equation 2 to obtain the solutions $(3, 3)$ and $(-3, -3)$. Who is correct? Explain your reasoning.

$x^2 + y^2 = 18$ Equation 1

$x - y = 0$ Equation 2

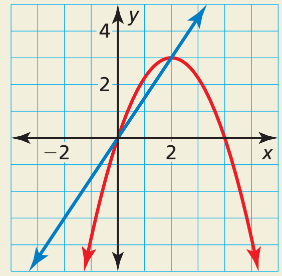
55. COMPARING METHODS Describe two different ways you could solve the quadratic equation. Which way do you prefer? Explain your reasoning.

$$-2x^2 + 12x - 17 = 2x^2 - 16x + 31$$

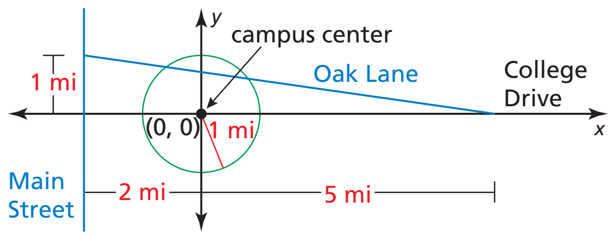
56. ANALYZING RELATIONSHIPS Suppose the graph of a line that passes through the origin intersects the graph of a circle with its center at the origin. When you know one of the points of intersection, explain how you can find the other point of intersection without performing any calculations.

57. WRITING Describe the possible solutions of a system that contains (a) one quadratic equation and one equation of a circle, and (b) two equations of circles. Sketch graphs to justify your answers.

58. HOW DO YOU SEE IT? The graph of a nonlinear system is shown. Estimate the solution(s). Then describe the transformation of the graph of the linear function that results in a system with no solution.



59. MODELING WITH MATHEMATICS To be eligible for a parking pass on a college campus, a student must live at least 1 mile from the campus center.



a. Write equations that represent the circle and Oak Lane.

b. Solve the system that consists of the equations in part (a).

c. For what length of Oak Lane are students *not* eligible for a parking pass?

60. CRITICAL THINKING Solve the system of three equations shown.

$$x^2 + y^2 = 4$$
$$2y = x^2 - 2x + 4$$
$$y = -x + 2$$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality. Graph the solution on a number line. *(Skills Review Handbook)*

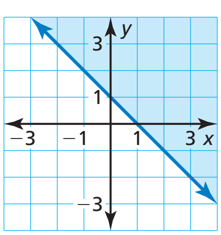
61. $4x - 4 > 8$

62. $-x + 7 \leq 4 - 2x$

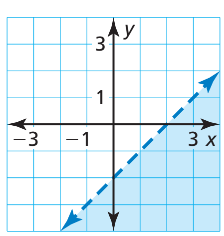
63. $-3(x - 4) \geq 24$

Write an inequality that represents the graph. *(Skills Review Handbook)*

64.



65.



66.